Black Holes in Asymptotically Safe Gravity

渐近安全引力中的黑洞

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In this chapter, we review the state-of-the-art of black holes in asymptotically safe gravity. After a brief recap of the asymptotic safety program, we shall summarize derivations and features of the main asymptotic safety-inspired models that have been constructed in the past by the so-called renormalization group improvement. Specifically, we will discuss static configurations, both in spherically and axially symmetric settings, the role played by the cosmological constant, and the impact of the collapse dynamics in determining black hole configurations realized in nature. In particular, we will review how quantum gravity could modify the Buchdahl limit and the corresponding conditions to form ultra-compact objects and Planckian black holes. We will then proceed by describing the most recent developments, particularly those aiming at making model building in asymptotic safety more rigorous and free from ambiguities. These include self-consistent and coordinate-independent versions of the renormalization group improvement and next steps to fill the gap between model building and renormalization group computations in asymptotic safety. Finally, we will focus on a selection of results that have been obtained from first-principle calculations or arguments, within and beyond asymptotic safety. Concretely, we will review the state-of-the-art in determining black hole entropy in asymptotic safety from a microstate counting and progress in deriving the quantum-corrected Newtonian potential. We will discuss how in quantum gravity theories linked to a gravitational path integral singularity resolution could be achieved by a dynamical suppression of singular configurations and how this is related to the form of the bare action. Finally, we will show that - independent of the specific ultraviolet completion of gravity - asymptotic modifications to Schwarzschild black holes are strongly constrained by the principle of least action at large-distance scales.

本章我们综述渐近安全引力中黑洞研究的最新进展。在简要回顾渐近安全纲领后，我们将总结过去通过所谓重整化群改进方法构建的、受渐近安全启发的主流模型的推导过程与核心特点。具体而言，我们将讨论球对称和轴对称情形下的静态构型，宇宙常数发挥的作用，以及坍缩动力学对确定自然界中实际存在的黑洞构型的影响。我们尤其会综述量子引力如何修正布赫达耳极限，以及形成超致密天体和普朗克黑洞的对应条件。随后我们将介绍该领域的最新进展，尤其是那些旨在让渐近安全框架下的模型构建更严谨、消除歧义的工作，其中包括重整化群改进的自洽、不依赖坐标的版本，以及填补渐近安全中模型构建与重整化群计算之间空白的下一步工作。最后，我们将聚焦于通过第一性原理计算或论证得到的、适用于渐近安全框架内外的部分代表性结果。具体来说，我们会综述渐近安全框架下通过微态计数确定黑洞熵的研究现状，以及推导量子修正牛顿势的研究进展。我们还会讨论，在与引力路径积分结合的量子引力理论中，如何通过动力学抑制奇异构型实现奇点消解，以及这和裸作用量的形式有何关联。最后我们将说明: 无论引力具体采用何种紫外完备方案，在大尺度上最小作用量原理都对史瓦西黑洞的渐近修正给出了极强约束。

Keywords

关键词

Quantum gravity · Renormalization group · Asymptotically safe gravity · Black holes

量子引力 · 重整化群 · 渐近安全引力 · 黑洞

Introduction

引言

Unraveling the quantum nature of black holes is among the most important objectives of research in quantum gravity. Despite the impressive achievements of Einstein's general relativity, the singularities and instabilities characterizing classical black holes indicate that a more fundamental description ought to take over the classical framework. How such a fundamental theory of gravity looks like is an outstanding open question.

揭示黑洞的量子本质是量子引力领域最重要的研究目标之一。尽管爱因斯坦的广义相对论取得了举世瞩目的成就，经典黑洞所特有的奇点与不稳定性表明，更基础的描述必然会取代经典框架。这种基础引力理论究竟以何种形式存在，目前仍是一个悬而未决的关键问题。

Over the years, several proposals have been put forth. Insofar as quantum gravity lives at extremely high energies - above the Planck scale - testing and discriminating between theories is challenging, and theoretical consistency has become a fundamental guidance in constraining different theories. Among the variety of consistency constraints, recovering a gravitational effective field theory in the infrared (IR) starting from the deep ultraviolet (UV) is arguably one of the most important requirements, and only a few theories have managed to pass this test so far. Among them, asymptotically safe gravity [1, 2] has emerged as a minimal while promising proposal, conjecturing that quantum gravity be described by a quantum field theory (QFT) whose UV behavior is controlled by an interacting fixed point of the gravitational renormalization group (RG) flow. The fixed point acts as an attractor for a subset of RG trajectories, providing a UV completion for the theory and rendering it renormalizable à la Wilson.

多年来，学界已经提出了多种理论方案。由于量子引力效应发生在普朗克能标以上的极高能区，对不同理论进行检验和区分难度极大，因此理论自洽性成为约束各类理论的核心指导原则。在众多自洽性约束中，从深紫外 (UV) 出发在红外 (IR) 区域得到引力有效场论，可以说是最重要的要求之一，目前仅有少数理论满足这一条件。渐近安全引力 [1, 2] 就是其中一个简洁且颇具前景的理论方案，该猜想认为量子引力可以用量子场论 (QFT) 描述，其紫外行为由引力重整化群 (RG) 流的相互作用不动点控制。该不动点是一部分 RG 轨迹的吸引子，为该理论提供了紫外完备性，使其符合威尔逊意义下的可重整化要求。

The presence of an asymptotically safe fixed point for quantum gravity, akin to the asymptotically free one in quantum chromodynamics, is responsible for a distinctive hallmark of asymptotic safety: its antiscreening character [3]. On the formal side, gravitational antiscreening is tied to the attractivity properties of the fixed point; on the phenomenological side, this hallmark can be encoded in an effective Newton coupling which vanishes in the regimes where quantum gravity effects are expected to be important. This intuitive picture has been extensively exploited in the literature to model deviations from general relativity induced by an asymptotically safe UV completion of gravity. The corresponding asymptotic safety-inspired models are typically obtained by replacing the observed Newton constant with an effective, coordinate dependent one, which smoothly interpolates between the observed value in the IR and its fixed-point scaling in the UV. This procedure is better known as RG improvement, and the resulting gravitational models are dubbed RG-improved spacetimes.

与量子色动力学中的渐近自由类似，量子引力存在渐近安全不动点这一特性，赋予了渐近安全一个标志性特征: 反屏蔽特性 [3]。在形式层面，引力反屏蔽与不动点的吸引性质直接相关；在唯象层面，这一特征可以体现为: 在量子引力效应本应显著的区域，有效牛顿耦合会趋于零。这一直观的图景已经在文献中被广泛用于建模渐近安全引力紫外完备性对广义相对论带来的修正。这类受渐近安全启发的模型通常的做法是，将观测到的牛顿常数替换为一个依赖坐标的有效牛顿常数，该常数在红外区域平滑过渡到观测值，在紫外区域则平滑过渡到不动点标度。这一过程被称为 RG 改进，得到的引力模型被称为 RG 改进时空。

The RG improvement has been a valuable instrument to explore possible implications of asymptotically safe gravity in astrophysics and cosmology, particularly at the dawn of asymptotically safe phenomenology. The method has even inspired an entirely new program, which is by now detached from asymptotically safe gravity, and goes under the name of ”scale-dependent gravity” [4-7]. Yet, more rigorous derivations and arguments, grounded either on the functional integral or on the effective action, are in order to unravel asymptotic safety-induced modifications of classical black holes and early-universe cosmology. Such fundamental approaches have been the focus of the asymptotic safety program in the past few years.

在渐近安全唯象学发展初期，RG 改进是探索渐近安全引力对天体物理和宇宙学可能影响的重要工具。该方法甚至催生了一个如今已经独立于渐近安全引力的全新研究方向，即“尺度依赖引力”[4-7]。然而，若要揭示渐近安全对经典黑洞和早期宇宙学带来的修正，仍需要基于泛函积分或有效行动的更严格推导和论证。这类基础研究在过去几年一直是渐近安全研究计划的核心方向。

In this chapter, we review of the state-of-the-art of black holes in asymptotically safe gravity (see also [8-10]), from the early works on RG-improved black holes to the most recent developments involving computations and considerations based on the gravitational effective action. In our narrative, we will mostly be following a chronological order.

在本章中，我们将综述渐近安全引力框架下黑洞研究的最新进展 (也可参见 [8-10])，涵盖从 RG 改进黑洞的早期工作到基于引力有效行动的计算与思考的最新发展。我们的叙述大致遵循时间顺序。

The chapter is organized as follows. In section ”Asymptotic Safety in a Nutshell” we will briefly review the asymptotic safety scenario for quantum gravity, providing all basic ingredients required for the understanding of the subsequent sections on asymptotically safe black holes. The ideas behind the RG improvement, its recipe, and its most pressing issues will be the topic of section ”RG Improvement: Key Idea.” We will discuss the resulting RG-improved black holes, in all their facets, in section ”RG-Improved Black Holes.” Attempts to ameliorating the method and removing its ambiguities will be the focus of section ”Improving the RG Improvement: Methods and Physical Results,” where we will also comment on the physical implications of these improvements and how they compare with past results. Section ”Toward Black Holes from First Principles” will be devoted to the subject of black holes from the effective action: we will discuss the most important developments of the past years, which unlocked intriguing aspects of black holes within and beyond asymptotic safety, grounded on first-principle calculations in quantum gravity. We will summarize all these points in our conclusions, section ”Summary and Conclusions.”

本章结构安排如下: 在“渐近安全概览”一节中，我们将简要回顾量子引力的渐近安全图景，给出理解后续渐近安全黑洞章节所需的全部基础内容。“RG 改进: 核心理念”一节将介绍 RG 改进背后的思想、实现方案以及当前最突出的问题。我们将在“RG 改进黑洞”一节全方位讨论由此得到的 RG 改进黑洞。“改进 RG 改进: 方法与物理结果”一节将聚焦改进方法、消除其不确定性的尝试，我们还会讨论这些改进的物理意义，以及与早期结果的对比。“从第一原理出发研究黑洞”一节专门讨论从有效行动出发研究黑洞的课题: 我们会介绍过去几年的核心进展，这些进展基于量子引力的第一原理计算，揭示了渐近安全内外黑洞值得关注的诸多新特性。我们将在最后一节“总结与结论”中对所有内容进行归纳。

Asymptotic Safety in a Nutshell

渐近安全简述

Asymptotically safe gravity [1, 2] is one of the most conservative approaches to quantum gravity. It relies on the framework of QFT and conjectures that the high-energy behavior of the gravitational RG trajectory realized by nature be controlled by a UV fixed point, where all (essential) running couplings approach a finite, nonzero value. This condition is known as ”asymptotic safety” and can be regarded as a non-perturbative generalization of the well-known concept of asymptotic freedom, whereby couplings vanish in the UV. Asymptotic freedom or safety guarantee that a theory be renormalizable, as well as UV complete with respect to a free or interacting fixed point, respectively. The first case can be related to perturbative renormalizability. The second one corresponds to a generalized notion of renormalizability, often regarded as ”non-perturbative renormalizability.” Power-counting arguments only hold for the former, while the existence of the latter cannot be determined a priori and must be investigated by appropriate RG techniques, typically beyond perturbation theory. One of their analytical realizations is the framework of the FRG [11], while corresponding lattice approaches are employed within the Euclidean and causal dynamical triangulation programs [12,13].

Thanks to these powerful methods, the asymptotic safety conjecture has been tested within a large number of approximations and against a variety of different starting assumptions (see [1, 2] and references therein). Notably, the asymptotic safety approach to quantum gravity does not need new physics, at least in principle, as long as its introduction is not required by compelling experimental evidence.

> 渐近安全引力 [1, 2] 是量子引力领域最保守的研究方案之一。它建立在量子场论框架之上，猜想自然界实现的引力重整化群 (RG) 轨迹的高能行为由一个紫外 (UV) 不动点控制，所有 (基本) 跑动耦合在该不动点趋近于有限非零值。这一条件被称为“渐近安全”，可视为著名的渐近自由概念的非微扰推广——渐近自由中耦合在紫外趋于零。渐近自由和渐近安全分别保证了理论可重整化，且各自对应自由不动点和相互作用不动点实现紫外完备。第一种情况对应微扰可重整性，第二种则对应广义的可重整性概念，常被称为“非微扰可重整性”。幂律计数论证仅适用于前者，后者是否存在无法先验确定，必须通过合适的重整化群技术 (通常超出微扰论范畴) 研究。这类方法中，一种解析实现是函数重整化群 (FRG) 框架 [11]，欧几里得动力学三角化方案和因果动力学三角化方案则采用了相应的格点方法 [12,13]。得益于这些有力的方法，渐近安全猜想已经在大量近似下、针对多种不同的初始假设得到了检验 (见 [1, 2] 及其中参考文献)。值得注意的是，渐近安全量子引力方案至少在原则上不需要引入新物理，除非确凿的实验证据要求引入新物理。

The FRG combines together two ingredients: on the one hand, the Wilsonian idea of renormalization [14] and, on the other hand, the functional approach introduced in QFT to perform a path integral quantization and to study scattering amplitudes. The result of this combination is a functional approach to renormalization, by which one can:

> 函数重整化群结合了两个核心要素: 一方面是威尔森的重整化思想 [14]，另一方面是量子场论中引入的、用于实现路径积分量子化和研究散射振幅的泛函方法。二者结合得到了重整化的泛函框架，通过该框架我们可以:

- Handle with any QFT, including those that are (perturbatively or power-counting) non-renormalizable but non-perturbatively renormalizable.

> - 处理任意量子场论，包括那些 (微扰或幂律计数意义下) 不可重整、但非微扰可重整的理论。

- Determine the bare action from first principles, i.e., as an RG fixed point.

> - 从第一性原理出发确定裸作用量，即将其确定为一个重整化群不动点。

- Derive the quantum effective action stemming from an RG fixed point and an appropriate number of initial conditions.

> - 推导出由一个重整化群不动点和适当数量初始条件得到的量子有效作用量。

To this end, one needs to introduce a scale-dependent version of the effective action, Γ_k , dubbed the "effective average action" (EAA). Here k is an artificial RG scale and not a physical momentum. It can be shown that the (Euclidean [1]) path integral can be translated into the following functional integro-differential equation for the effective average action

为此，我们需要引入一个依赖能标的有效作用量形式 Γ_k，称为“有效平均作用量”(EAA)。此处 k 是人工引入的重整化群能标，而非物理动量。可以证明，(欧几里得 [1]) 路径积分可以转化为如下关于有效平均作用量的泛函积分微分方程

$$k\partial_k\Gamma_k = \frac{1}{2}\,\mathrm{STr}\left(\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1} k\partial_k\mathcal{R}_k\right), \tag{1}$$

better known as the Wetterich equation [28,29]. Here $\mathcal{R}_k \propto k^2$ is a RG scale-dependent mass term which is added to the original Lagrangian to regularize the path integral and to integrate quantum fluctuations with momenta $p^2 \gtrsim k^2$. The symbol ”STr” stands for a supertrace, summing over internal indices and integrating over the volume in coordinate or momentum space. Finally, $\Gamma_k^{(2)}$ denotes the second functional derivative of the EAA with respect to all fields appearing in Γ_k .

它更广为人知的名称是维特里希方程 [28,29]。此处 $\mathcal{R}_k \propto k^2$ 是依赖重整化群能标的质量项，被添加到原本的拉格朗日中以正则化路径积分，对动量满足 $p^2 \gtrsim k^2$ 的量子涨落完成积分。符号“STr”代表超迹，即对内部指标求和，并在坐标空间或动量空间对体积积分。最后，$\Gamma_k^{(2)}$ 代表有效平均作用量对出现在 Γ_k 中所有场的二阶泛函导数。

The RG fixed points are identified by the conditions $k\partial_k g_i(k) = 0$, where the index i labels the running couplings in Γ_k and $g_i(k)$ denotes their dimensionless counterpart. A fixed point can be reached by some RG trajectories either in the IR or in the UV. Accordingly, a given fixed point provides a UV completion for all RG trajectories belonging to its basin of attraction. The dimension of the latter determines the number N of relevant directions and thus the number of free parameters of the theory. Note that this number depends on the specific fixed point considered.

重整化群不动点由条件 $k\partial_k g_i(k) = 0$ 确定，其中指标 i 标记 Γ_k 中的跑动耦合，$g_i(k)$ 表示对应无量纲耦合。RG 轨迹可以在红外 (IR) 或紫外方向趋近某个不动点。因此，一个给定不动点会为其吸引域内所有 RG 轨迹提供紫外完备。吸引域的维数决定了相关方向的数量 N，也就决定了理论自由参数的数量。注意该数量取决于所考虑的具体不动点。

Once one identifies the set of fixed points, the next question to ask is whether any of the RG trajectories departing from a fixed point in the UV can reach an IR that is compatible with experimental and observational data. This can be verified by integrating the flow equation (1) complemented by a sufficient number of initial conditions dictated by observations. If a solution compatible with these initial conditions exists, and if it reaches a fixed point in the UV, then:

在确定所有不动点集合后，下一个问题是: 紫外端从某一不动点出发的 RG 轨迹，最终能否得到和实验观测数据一致的红外结果。这可以通过积分流方程 (1) 来验证，积分需要补充由观测给出的足够数量初始条件。如果满足这些初始条件的解存在，且该解在紫外趋近一个不动点，那么:

- The RG trajectory Γ_k^{sol} realized by the particular system analyzed is consistent at all scales, and in particular, it is UV complete.

- 所分析特定系统实现的 RG 轨迹 Γ_k^{sol} 在所有尺度下都是自洽的，且它尤其满足紫外完备性。

- The theory associated with the RG trajectory Γ_k^{sol} is renormalizable.

- 与该 RG 轨迹 Γ_k^{sol} 关联的理论是可重整化的。

- Its observables, including scattering amplitudes, can be computed using the standard effective action, which is obtained as the limit $\Gamma_0 \equiv \lim_{k\to 0} \Gamma_k^{\text{sol}}$.

- 包括散射振幅在内，它的所有可观测量都可以通过标准有效作用量计算得到，该有效作用量就是极限 $\Gamma_0 \equiv \lim_{k\to 0} \Gamma_k^{\text{sol}}$ 。

- If the theory Γ_k^{sol} is UV completed by a fixed point with N relevant directions, all infinitely many couplings in the corresponding effective action Γ_0 will be written in terms of N free parameters only.

- 若理论 Γ_k^{sol} 被一个拥有 N 个相关方向的不动点实现紫外完备，则对应有效作用量 Γ_0 中无穷多个耦合都可以仅用 N 个自由参数表示。

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[1] FRG computations are typically performed in Euclidean. First steps toward Lorentzian calculations have been taken in [15-24], while Lorentzian computations based on the spectral FRG have been developed and applied in [25-27].

[1] FRG 计算通常在欧几里得符号下进行。洛伦兹符号下计算的初步工作已完成于文献 [15-24]，而基于谱 FRG 的洛伦兹计算则在文献 [25-27] 中得到了发展与应用。

The next ingredient to discuss is the practical resolution of the Wetterich equation. Albeit this is exact and provides a clear recipe to compute the functional integral, solving it is involved, and one has to resort to approximations. In particular, independent of the specific theory - identified by a set of fields and the symmetries of their interactions - the functional Γ_k contains infinitely many interaction terms. In order to make computations doable, one possibility is to project the RG flow onto a manageable subspace of couplings. The calculation can then be improved in a step-by-step fashion, exploring larger and larger subspaces. In practice, this projection is performed by expanding Γ_k according to a certain criterion (e.g., a derivative or vertex expansion) and by ”truncating” the resulting series to a certain order. Obtaining the same results independent of the type of expansion and truncation order is considered as evidence of their stability [30].

接下来要讨论的内容是 Wetterich 方程的实际求解。尽管该方程是精确的，且给出了计算泛函积分的清晰方案，但求解它十分复杂，必须借助近似方法。具体而言，无论由一组场及其相互作用对称性定义的具体理论是什么，该泛函 Γ_k 都包含无穷多个相互作用项。为了让计算可以进行，一种可行方案是将 RG 流投影到一个可处理的耦合子空间。随后可以逐步改进计算，探索越来越大的子空间。实际操作中，这种投影通过以下方式实现: 按照一定判据展开 Γ_k (例如导数展开或顶点展开)，再将展开得到的级数“截断”到某一阶。结果不依赖展开类型和截断阶数被认为是结果稳定的证据 [30]。

In the case of gravity, a particularly convenient way to express Γ_k is via a curvature expansion of the form [31]

对于引力，表达 Γ_k 一种格外方便的方式是形如 [31] 的曲率展开

$$\Gamma_k = \int d^4x\sqrt{-g}\left(\frac{1}{16\pi G_k}(R-2\Lambda_k) + R g_{R,k}(\Box)R + C_{\mu\upsilon\sigma\rho} g_{C,k}(\Box)C^{\mu\upsilon\sigma\rho}\right).$$

(2)

Here $\Box = g_{\mu\nu}D^\mu D^\nu$ is the d' Alembertian operator, and $G_k, \Lambda_k, g_{R,k}(\Box)$, and $g_{C,k}(\Box)$ are the running Newton, cosmological, and quartic couplings, respectively. Only the latter two couplings can depend on the d' Alembertian operator, since in the volume term $\Box$ cannot act on any field, while a term $F_k(\Box)R/G_k$ is equivalent to R/G_k modulo a total derivative. Provided that a well-definite UV completion exists, the limit $k \to 0$ defines an effective action resembling closely the structure in Eq. (2) [31]

此处 $\Box = g_{\mu\nu}D^\mu D^\nu$ 是达朗贝尔算符，$G_k, \Lambda_k, g_{R,k}(\Box)$ 和 $g_{C,k}(\Box)$ 分别是跑动牛顿耦合、宇宙学耦合和四次耦合。仅后两项耦合可以依赖达朗贝尔算符，因为在体积项中 $\Box$ 无法作用在任何场上，而项 $F_k(\Box)R/G_k$ 模全导数等价于 R/G_k 。若存在定义良好的紫外完备，则极限 $k \to 0$ 定义的有效作用量与式 (2) 的结构非常相似 [31]

$$\Gamma_0 = \int d^4x\sqrt{-g}\left(\frac{1}{16\pi G_N}(R-2\Lambda) + R g_R(\Box)R + C_{\mu\upsilon\sigma\rho} g_C(\Box)C^{\mu\upsilon\sigma\rho}\right),$$

(3)

where $G_N \equiv G_0$ and $\Lambda \equiv \Lambda_0$ are the observed Newton and cosmological constants. The "form factors" $g_R(\Box)$ and $g_C(\Box)$ encode instead the physical momentum dependence of the quartic couplings [24, 25, 31-35], generalizing the relation $\partial^2 \sim -p^2$ to curved spacetimes [31]. As the effective action (3) is the result of integrating over all quantum fluctuations, quantities computed using it already encode all quantum effects. In particular, quantum black holes and cosmologies ought to be derived as solutions to the dressed field equations:

其中 $G_N \equiv G_0$ 和 $\Lambda \equiv \Lambda_0$ 是观测得到的牛顿常数和宇宙学常数。而“形状因子” $g_R(\Box)$ 和 $g_C(\Box)$ 则编码了四次耦合的物理动量依赖 [24, 25, 31-35]，将关系 $\partial^2 \sim -p^2$ 推广到了弯曲时空 [31]。由于有效作用量 (3) 是对所有量子涨落积分的结果，用它计算得到的量已经包含了所有量子效应。具体来说，量子黑洞和量子宇宙学都应当作为修饰后的场方程的解导出:

$$\frac{\delta\Gamma_0\left[g^{sol}_{\mu\upsilon}\right]}{\delta g^{sol}_{\mu\upsilon}} = 0. \tag{4}$$

We now have all ingredients to introduce the so-called RG improvement and then to step into the main topic of this chapter.

现在我们已经掌握了所有准备内容，可以引入所谓的 RG 改进，进而进入本章的主题。

RG Improvement: Key Idea

RG 改进: 核心思想

The RG improvement was introduced in the context of gauge and matter field theories as a shortcut to access some of terms in the effective action and, in some cases, determine leading-order modifications to the solutions to the corresponding effective field equations [36-40].

RG 改进最初是在规范场与物质场理论中提出的，作为获取有效作用量中某些项的捷径，在部分场景中还能确定对应有效场方程解的领头阶修正 [36-40]。

The RG improvement procedure consists of the following steps. First, starting from a classical system, e.g., an action or a solution, one replaces its couplings with their running counterparts, $g_i \rightarrow g_i(k)$. At the level of the action, this step is akin to promoting an ansatz for the action to an EAA Γ_k . Second, the running couplings $g_i(k)$ are replaced with the solutions to the corresponding RG equations, $k\partial_k g_i(k) = \beta_i\left[g_j(k)\right]$, complemented by suitable physical initial conditions. Finally, k is identified with a scale of the system which could act as a physical IR cutoff.

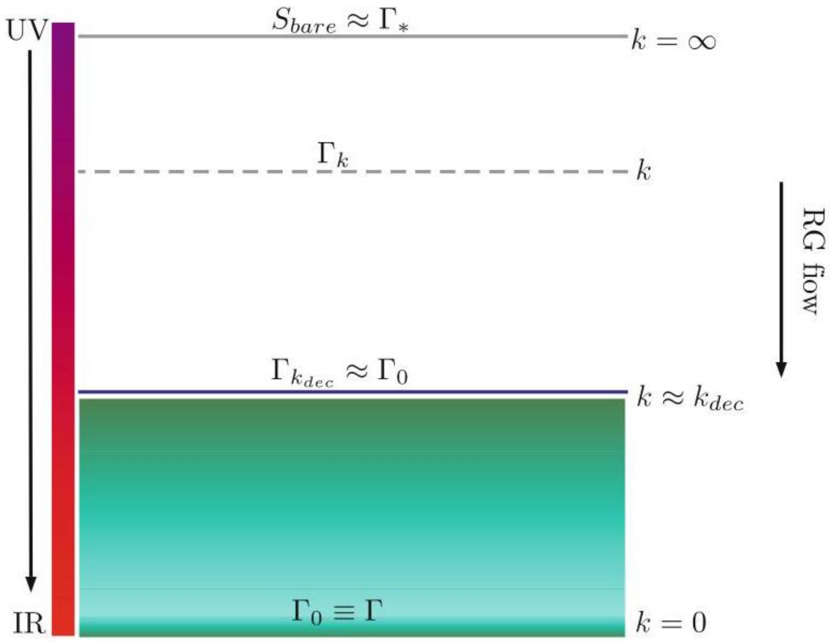
RG 改进流程包含以下步骤: 首先，从经典系统 (例如作用量或解) 出发，将其耦合替换为跑动耦合 $g_i \rightarrow g_i(k)$ 。在作用量层面，这一步类似于将作用量的拟设提升为有效平均作用量 EAA Γ_k 。其次，将跑动耦合 $g_i(k)$ 替换为对应 RG 方程的解 $k\partial_k g_i(k) = \beta_i\left[g_j(k)\right]$ ，并辅以合适的物理初始条件。最后，将 k 确定为系统的能标，它可以作为物理红外截断。

The reason why this procedure is supposed to work, at least in simple cases, lies in the so-called decoupling mechanism [41]: if in the flow of Γ_k there are physical IR scales (e.g., masses, curvature, or interactions terms) that prevail over the unphysical regulator $\mathcal{R}_k$ below a certain threshold scale $k_{\text{dec}} -$ dubbed the decoupling scale - then the right-hand side of Eq. (1) gets smaller, thus slowing down the RG flow; as a result, the EAA at the decoupling scale approximates the full effective action Γ_0 . This idea is illustrated in Fig. 1. In particular, the decoupling mechanism can give access to some of the interaction terms in the effective action that were not considered in the initial truncation. This is the case, for instance, in scalar electrodynamics, where the decoupling condition together with the RG improvement can be used to determine the Coleman-Weinberg effective potential [36, 41] .

该流程至少在简单场景下能够成立，原因在于所谓的退耦机制 [41]: 如果 Γ_k 的流中存在物理红外标度 (例如质量、曲率或相互作用项)，其在某一阈值能标 (称为退耦能标 $k_{\text{dec}} -$) 以下优于非物理调节器 $\mathcal{R}_k$ ，那么式 (1) 的右侧会变小，从而减慢 RG 流；最终，退耦能标处的 EAA 近似等于全有效作用量 Γ_0 。这一思想如图 1 所示。特别地，通过退耦机制可以得到初始截断中未考虑的有效作用量的部分相互作用项。例如标量电动力学中就是如此，结合退耦条件与 RG 改进可以确定科尔曼-温伯格有效势 [36, 41] 。

Fig. 1 Pictorial representation of the decoupling mechanism [42]. If one or more physical IR scales in the scale-dependent effective action Γ_k overcomes the artificial regulator $\mathcal{R}_k$ in Eq. (1), the flow slows down and eventually freezes out, so that $\Gamma_{k_{\text{dec}}} \approx \Gamma_0$

图 1 退耦机制的图示 [42]。如果依赖能标的有效作用量 Γ_k 中的一个或多个物理红外标度超过了式 (1) 中的人工调节器 $\mathcal{R}_k$ ，RG 流会减慢并最终冻结，因此 $\Gamma_{k_{\text{dec}}} \approx \Gamma_0$



Grounded on the decoupling mechanism, the RG improvement procedure might allow to capture some of the leading-order quantum corrections to classical systems. Yet, its application to gravity is far from straightforward.

基于退耦机制，RG 改进流程可以得到经典系统的部分领头阶量子修正。但它在引力中的应用远非易事。

Firstly, classical gravitational systems or phenomena are typically characterized by a number of competing physical IR scales, making the "cutoff identification" more involved than in the case of standard quantum field theories, where one can naïvely relate a momentum scale with the inverse of the radial distance. Specifically, despite much effort [41,43-47], finding a clear and unique way to identify the cutoff is not straightforward and has become a source of ambiguities.

首先，经典引力系统或现象通常存在多个相互竞争的物理红外标度，这使得“截断确定”比标准量子场论的情况更复杂——标准量子场论中可以直接将动量标度与径向距离的倒数关联。具体而言，尽管已有诸多研究工作 [41,43-47]，找到清晰唯一的截断确定方法仍非易事，这已经成为歧义的来源之一。

Secondly, the implementation of the RG improvement can in principle be performed at the level of the action, which should be more natural in relation with the decoupling mechanism or at the level of the field equations or solutions, which is technically less involved and more direct. It is clear however that these procedures can be inequivalent, since an application at the level of the action would yield more terms in the field equations. This is thereby a second source of ambiguity.

其次，RG 改进原则上既可以在作用量层面实现 (这相对于退耦机制而言更自然)，也可以在场方程或解的层面实现 (后者技术上更简单直接)。但很明显这两种流程并不等价，因为在作用量层面实施会在场方程中产生更多项。这便是歧义的第二个来源。

Thirdly, in the context of gravity, there are two more potential issues: back-reaction effects, due to the presence of a dynamical and fluctuating metrics that have to replace the classical, typically singular backgrounds of general relativity, and coordinate independence, which might be lost if the RG improvement is not carefully applied.

第三，在引力场景中还存在另外两个潜在问题: 反作用效应，它源于需要用动力学涨落度量替换广义相对论中经典的、通常奇异的背景；以及坐标不变性，如果 RG 改进应用不当就可能失去坐标不变性。

In spite of the aforementioned issues, the RG improvement has been a powerful tool to build asymptotic safety-inspired models and explore possible signatures and consequences of quantum gravity. A selection of these results in black hole physics is the topic of section ”RG-Improved Black Holes.” Some recent developments aiming at solving the ambiguities and problems of the RG improvement will be discussed in section ”Improving the RG Improvement: Methods and Physical Results.” Finally, in section ”Toward Black Holes from First Principles” we will report on some findings on quantum black holes stemming from first-principle computations within and beyond asymptotically safe gravity.

尽管存在上述问题，RG 改进仍是构建渐近安全启发模型、探索量子引力可能特征与结论的有力工具。本文“RG 改进黑洞”一节将介绍该方法在黑洞物理中取得的部分代表性结果。本文“改进 RG 改进: 方法与物理结果”一节会讨论解决 RG 改进现有歧义与问题的最新进展。最后，在“从第一原理出发研究黑洞”一节，我们将介绍渐近安全引力内外基于第一原理计算得到的量子黑洞相关研究结果。

RG-Improved Black Holes

RG 改进黑洞

In this section, we review a selection of asymptotic safety-inspired black hole models that have been constructed in the past years via the RG improvement procedure. We will start from the case of spherically symmetric, asymptotically flat RG-improved black holes (Section ”The Spherically Symmetric, Asymptotically Flat Case”). These are the simplest RG-improved models and chronologically the first ones that have been developed. Next, we will discuss the role of the cosmological constant (Section ”The Role of the Cosmological Constant”) and the generalization to axially symmetric systems (Section ”Rotating RG-Improved Black Holes and Their Shadows”). We will proceed with a discussion of dynamical RG-improved models, describing the formation of quantum black holes from a gravitational collapse, and we will conclude by detailing how the classical Buchdahl limit could be affected by gravitational antiscreening (Section ”Gravitational Collapse and Improved Buchdahl Limit”).

在本节中，我们将回顾近年来通过 RG 改进方法构建的、受渐近安全启发的精选黑洞模型。我们将从球对称渐近平直 RG 改进黑洞的情况开始 (章节「球对称渐近平直情形」)。这些是最简单的 RG 改进模型，也是按时间顺序最早开发的模型。接下来，我们将讨论宇宙学常数的作用 (章节「宇宙学常数的作用」)，以及对轴对称系统的推广 (章节「旋转 RG 改进黑洞及其阴影」)。我们之后会讨论动力学 RG 改进模型，描述引力坍缩过程中量子黑洞的形成，最后我们会详细阐述经典布赫达极限会如何受到引力反屏蔽的影响 (章节「引力坍缩与改进布赫达极限」)。

The Spherically Symmetric, Asymptotically Flat Case

球对称渐近平直情形

This subsection summarizes the works that pioneered the study of RG-improved black holes [48-50] and their generalizations in the presence of extra dimensions [51,52]. The focus is the case of static, spherically symmetric, asymptotically flat black holes. The RG improvement of the same system and its generalization to the charged case were also subsequently considered in [5, 53-59], whereby the use of different cutoff identifications leads to results in qualitative agreement with [48-50]. Recent developments based on more rigorous versions of the RG improvement and first-principle calculations will be discussed in sections "Improving the RG Improvement: Methods and Physical Results" and "Toward Black Holes from First Principles," respectively.

本小节总结了开创 RG 改进黑洞研究的工作 [48-50]，以及它们在额外维存在下的推广 [51,52]。本文聚焦于静态、球对称、渐近平直黑洞的情形。同一体系的 RG 改进及其向带电情形的推广随后也在文献 [5,53-59] 中被研究，这些工作使用了不同的截断标识，得到的结果与 [48-50] 在定性上一致。基于更严谨的 RG 改进版本的最新进展，以及第一性原理计算，将分别在“改进 RG 改进: 方法与物理结果”和“通向第一性原理构造的黑洞”章节中讨论。

Bonanno-Reuter Black Holes [48,49]

博南诺-罗伊特黑洞 [48,49]

Static, spherically symmetric, asymptotically flat black holes yield an ideal playground to develop and investigate spacetime models beyond general relativity. In their most commonly studied incarnation, the time and radial metric components in Schwarzschild coordinates are assumed to be inversely related, $g_{rr} = g_{tt}^{-1}$, and the line element reads

静态球对称渐近平直黑洞是拓展研究广义相对论之外时空模型的理想测试平台。在最广为研究的模型设定中，史瓦西坐标下时间分量和径向的度规分量满足反比关系，即 $g_{rr} = g_{tt}^{-1}$，线元形式为

$$ds^2 = -f(r)\,dt^2 + f^{-1}(r)\,dr^2 + r^2 d\Omega^2. \tag{5}$$

Inasmuch classical spacetimes in general relativity must satisfy the vacuum Einstein field equations, the classical lapse function $f = f_{cl}$ takes the form

广义相对论中的经典时空必须满足真空爱因斯坦场方程，因此经典时移函数 $f = f_{cl}$ 形式为

$$f_{cl}(r) = 1 - \frac{2mG_0}{r}, \tag{6}$$

where m is the mass of the black hole and $G_0 \equiv G_N$ is the observed value of the Newton coupling. Assuming that black holes realized by nature are such that $g_{rr} = g_{tt}^{-1}$, deviations from general relativity can be embedded in an effective Newton coupling $G(r)$ modifying the radial dependence of the classical lapse function:

其中 m 为黑洞质量, $G_0 \equiv G_N$ 是牛顿引力耦合的观测值。假设自然中真实存在的黑洞满足 $g_{rr} = g_{tt}^{-1}$ ，对广义相对论的偏离可以被纳入有效牛顿耦合 $G(r)$ 中，从而修改经典时移函数的径向依赖关系:

$$f_{qu}(r) = 1 - \frac{2mG(r)}{r}. \tag{7}$$

In the following we shall analyze the consequences an effective Newton coupling originating from the RG improvement of the classical metric, following the original derivation in [48, 49] .

接下来我们将遵循 [48, 49] 中的原始推导，分析由经典度规经 RG 改进得到的有效牛顿耦合带来的结论。

According to the recipe detailed in section ”RG Improvement: Key Idea,” the RG improvement of a classical metric involves replacing the Newton constant with its running counterpart:

根据“RG 改进: 核心理念”一节给出的方法，对经典度规做 RG 改进需要将牛顿常数替换为它的跑动形式:

$$G_0 \to G_k \tag{8}$$

The functional dependence of G_k on the RG scale k is dictated by the beta function of its dimensionless version, $g_k = G_k k^2$. In an approximation where all other couplings in the action vanish, the running is given by [48]

G_k 对 RG 标度 k 的函数依赖由其无量纲形式 $g_k = G_k k^2$ 的 β 函数决定。在作用量中所有其他耦合都为零的近似下，跑动关系由 [48] 给出

$$G_k = \frac{G_0}{1 + g_*^{-1} G_0 k^2}, \tag{9}$$

with g_* being the fixed-point value of the dimensionless Newton coupling g_k . The running of the two versions of the Newton coupling is displayed in Fig. 2.

其中 g_* 是无量纲牛顿耦合 g_k 的不动点值。两种形式牛顿耦合的跑动行为如图 2 所示。

Motivated by the case of quantum field theories on flat spacetimes, whereby the Wilsonian IR momentum is inversely related to the radial coordinate, $k \sim 1/r$, in [48, 49] Bonanno and Reuter set the scale via the proper distance between the origin $r = 0$ and a generic point along a purely radial geodesic, $k \sim \xi/D(r)$, with ξ a numerical factor (presumably of $\mathcal{O}(1)$) setting the scale of quantum gravity, and the geodesic distance given by

受平直时空量子场论结论启发——该框架威尔森红外动量与径向坐标成反比，即 $k \sim 1/r$ ，博南诺和罗伊特在 [48, 49] 中通过原点 $r = 0$ 与纯径向测线上任意一点之间的固有时距离来设定标度，即 $k \sim \xi/D(r)$ ，其中 ξ 是设定量子引力标度的数值因子 (通常认为约为 $\mathcal{O}(1)$)，测地线距离由下式给出

$$D(r) = \int_0^r dr' |f_{cl}(r')|^{-1/2}. \tag{10}$$

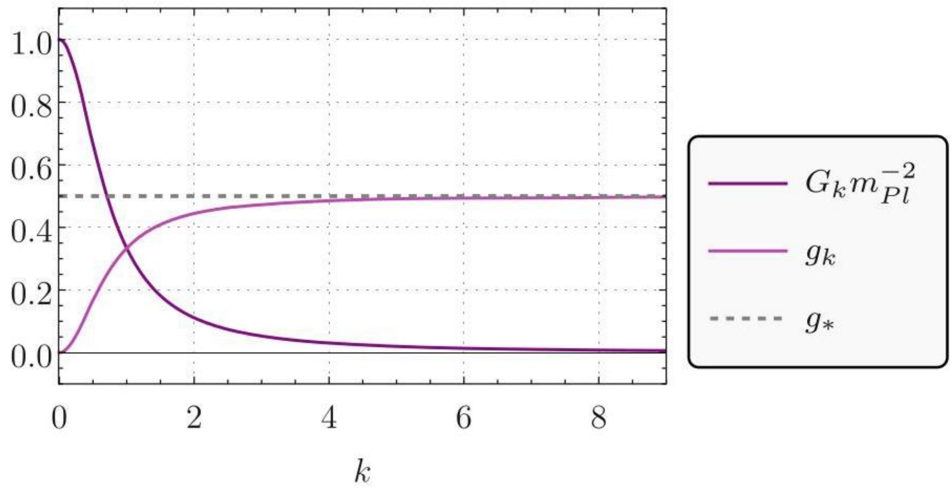


Fig. 2 Running of the dimensionful (purple line) and dimensionless (magenta line) Newton couplings with the RG scale k , according to Eq. (9), with $g_* = 1/2$ (gray, dashed line). The dimensionful Newton coupling matches its IR value for $k = 0$ and vanishes in the UV due to the fixed-point scaling, $G_k \sim g_* k^{-2}$. By contrast, its dimensionless counterpart is zero in the IR and approaches its fixed-point value g_* as $k \to \infty$

图 2 根据式 (9)，有量纲牛顿耦合 (紫色线) 和无量纲牛顿耦合 (品红色线) 随 RG 标度 k 的跑动行为，其中 $g_* = 1/2$ 为灰色虚线。有量纲牛顿耦合在 $k = 0$ 处与红外值一致，由于不动点标度行为在紫外区趋于零，即 $G_k \sim g_* k^{-2}$ 。与之相对，它的无量纲对应量在红外区为零，当 $k \to \infty$ 时趋近于不动点值 g_*

It is important to notice that this is only an approximation, as the definition of the proper distance (or any other physical scale) in terms of the classical metric is not self-consistent (cf. section ”Self-Consistency and Iterative RG Improvement”).

需要注意的是这只是近似，用经典度规定义固有距离 (或任何其他物理标度) 并不自洽 (参见“自洽性与迭代 RG 改进”一节)。

Due to a discontinuity at the classical Schwarzschild radius, $r = r_s = 2mG_0$, the analytic form of the proper distance depends on whether r is smaller or bigger than the Schwarzschild radius. Close to the classical singularity, it behaves as

由于经典史瓦西半径 $r = r_s = 2mG_0$ 处存在不连续性，固有距离的解析形式取决于 r 比史瓦西半径更大还是更小。在经典奇点附近，其行为为

$$D_{r \ll l_{Pl}}(r) \sim \frac{2}{3} \frac{r^{3/2}}{\sqrt{2mG_0}} (1 + O(r)), \tag{11}$$

whereas at large distances, it scales as

而在远距离处，它的标度行为为

$$D_{r \gg l_{Pl}}(r) \sim r + O\left(r^0\right). \tag{12}$$

In order to perform analytical calculations, one can exploit an approximate interpolating function

为了进行解析计算，可以利用近似插值函数

$$D(r) \approx \sqrt{\frac{2r^3}{2r + 9mG_0}}, \tag{13}$$

that smoothly connects the aforementioned asymptotic behaviors. The RG improvement procedure thereby results in a new black hole whose lapse function reads

它可以平滑连接上述渐近行为。RG 改进流程最终得到一个新黑洞，其时移函数为

$$f_{qu}(r) = 1 - \frac{2mG(r)}{r} = 1 - \frac{4G_0 mr^2}{2r^3 + g_*^{-1}\xi^2 G_0 (2r + 9mG_0)}. \tag{14}$$

Setting $g_*^{-1}\xi^2 = 41/(10\pi)$,[2] the asymptotic behavior of the time component of the metric coincides with the leading-order corrections to the Newtonian potential [61- 63]. [3] In the opposite regime, close to the would-be singularity at $r = 0$, Bonanno-Reuter black holes behave like the most commonly studied regular black holes: they have a de Sitter core with effective cosmological constant:

Setting $g_*^{-1}\xi^2 = 41/(10\pi)$,[2] 处，度量时间分量的渐近行为与牛顿势的领头阶修正一致 [61-63]。[3] 在相反的 regime 中，在靠近原奇点 $r = 0$ 的位置，博纳诺-罗伊特黑洞的行为和最常见的正则黑洞一致: 它们拥有带有效宇宙学常数的德西特核心:

$$\Lambda_{\text{eff}} = \frac{4}{3g_*^{-1}\xi^2 G_0}. \tag{15}$$

As is typical for regular black holes, Bonanno-Reuter spacetimes can display two, one, or no horizons, depending on the value of the mass in Planck units (cf. Fig. 3).

和正则黑洞的典型特征一致，博纳诺-罗伊特时空可根据普朗克单位下的质量取值，呈现两个、一个或零个视界 (参见图 3)。

The causal structure of spacetime is similar to that of a classical Reissner-Nordström black hole. A test particle departing from the region I ($r < \infty$) will cross the outer horizon r_+ , thus entering the zone enclosed by the two horizons (region II). Due to the inverse sign of the time component of the metric, particles' geodesics proceed toward the inner horizon r_- and cross it, reaching region III. At this point, however, the sign of the lapse function becomes negative again, and geodesics cross back the inner horizon, in the opposite direction, and land in region IV (see Penrose diagram in Fig. 4).

该时空的因果结构和经典的莱斯纳-诺德斯特罗姆黑洞类似。一个从区域 I ($r < \infty$) 出发的测试粒子会穿过外视界 r_+，进入两个视界包围的区域 (区域 II)。由于度量时间分量的符号反转，粒子的测地线会朝向内视界 r_- 运动并穿过它，到达区域 III。但在这一点，lapse 函数的符号会再次反转，测地线会反向穿回内视界，进入区域 IV(参见图 4 中的彭罗斯图)。

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[2] The value originally used in [49] was $g_*^{-1}\xi^2 = 118/(15\pi)$ and was based on the corrections to the Newtonian potential computed in [60].

[2] 文献 [49] 最初使用的取值为 $g_*^{-1}\xi^2 = 118/(15\pi)$，该取值基于文献 [60] 中计算得到的牛顿势修正。

[3] Such an identification is however a stretch, as we will discuss in section "Toward a Dressed Newtonian Potential Within Asymptotic Safety".

[3] 不过正如我们将在“渐近安全框架内修整后的牛顿势”一节讨论的，这样的定义其实比较牵强。

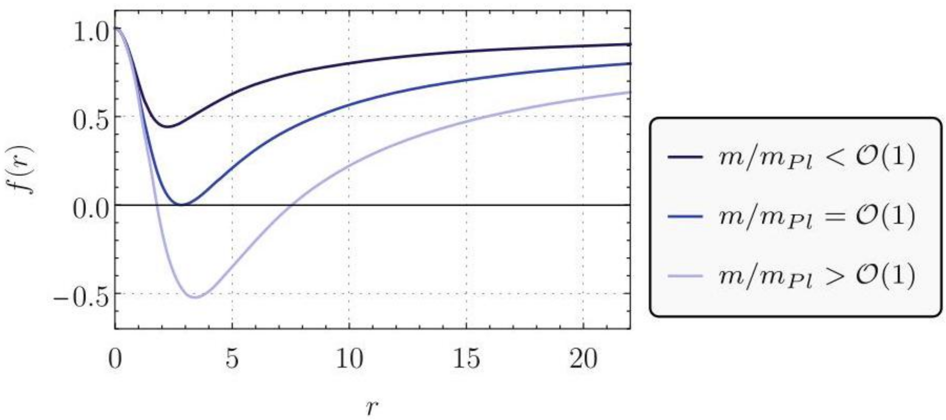

Fig. 3 RG-improved lapse function (14) for different values of the black hole mass in Planck units. Above the critical value $m/m_{Pl} \sim 1$ the spacetime has two horizons. If m is decreased, the two horizons get closer, until they merge into a single horizon for $m \sim m_{Pl}$. Below this threshold value, the two horizons disappear, since the roots $r_\pm$ of the Bonanno-Reuter lapse function become complex conjugate

图 3 不同普朗克单位黑洞质量取值下, 由 RG 改进得到的 lapse 函数 (14)。当质量高于临界值 $m/m_{Pl} \sim 1$ 时，时空拥有两个视界。若 m 减小，两个视界会逐渐靠近，最终在质量取 $m \sim m_{Pl}$ 时合并为一个视界。当质量低于该阈值时，两个视界会消失，因为博纳诺-罗伊特 lapse 函数的根 $r_\pm$ 变为复共轭根

In spite of a similar causal structure, gravitational antiscreening could impact the phenomenon of mass inflation. Classically, due to influx and outflux of gravitational waves during the collapse of a star, the source term for the variation of the mass increases and blows up at the Cauchy horizon, where the spacetime develops a null singularity. The antiscreening of gravity at high energies can in principle limit the increase rate of the mass function and could at least weaken the singularity classically formed at the Cauchy horizon [48]. However, the question of mass inflation and inner horizon (in)stability is still under heated debate [64-71].

尽管因果结构相似，但引力反屏蔽会影响质量暴涨现象。经典情况下，恒星坍缩过程中由于引力波的流入和流出，质量变化的源项会不断增大，并在柯西视界处发散，时空会在柯西视界形成类空奇点。高能下的引力反屏蔽原则上可以限制质量函数的增长率，至少可以弱化经典情况下在柯西视界形成的奇点 [48]。但目前关于质量暴涨以及内视界 (不) 稳定性的问题仍然存在激烈争论 [64-71]。

Evaporation of Bonanno-Reuter Black Holes [50]

博南诺-罗伊特黑洞的蒸发 [50]

The evaporation of Bonanno-Reuter black holes has been studied in detail by the same authors in [50], although some features could be guessed by the static limit of the previous section.

尽管部分特征可以通过上一节的静态极限推测，但原作者已经在文献 [50] 中详细研究了博南诺-罗伊特黑洞的蒸发。

In general, a key element in determining the endpoint of the evaporation process of a (classical or quantum) black hole is its temperature - defined as the surface gravity at the outer horizon, r_+ . In the case of Bonanno-Reuter black holes, it reads

一般而言，决定 (经典或量子) 黑洞蒸发过程终点的核心要素是黑洞温度——它定义为外视界处的表面引力，r_+ 。对于博南诺-罗伊特黑洞，温度可写为

$$T = \frac{1}{4\pi}\frac{\partial f_{qu}(r)}{\partial r}\bigg|_{r=r_+} = \frac{1}{8\pi}\frac{mG_0 r_+\left(r_+^3 - g_*^{-1}G_0 r_+ - g_*^{-1}G_0^2 9m\right)}{\left(r_+^3 + g_*^{-1}\left(r_+ + 9/2G_0 m\right)\right)^2}. \tag{16}$$

Starting from the sub-critical configuration with two horizons, the evaporation process makes the temperature increase as m decreases, in analogy with the classical case. However, at variance of classical evaporating black holes, where $T \propto 1/m$ for all values of the black hole mass, the temperature of Bonanno-Reuter black holes decreases and eventually vanishes as the two horizons collapse into one (see Fig. 5). The evaporation process thus stops, leading to a Planckian black hole remnant.

从带有两个视界的亚临界构型出发，和经典情况类似，蒸发过程会让温度随 m 减小而升高。但和经典蒸发黑洞不同，经典黑洞中 $T \propto 1/m$ 对所有黑洞质量都成立，而博南诺-罗伊特黑洞在两个视界坍缩为一时温度会下降并最终消失 (见图 5)。因此蒸发过程会停止，留下一个普朗克黑洞残余。

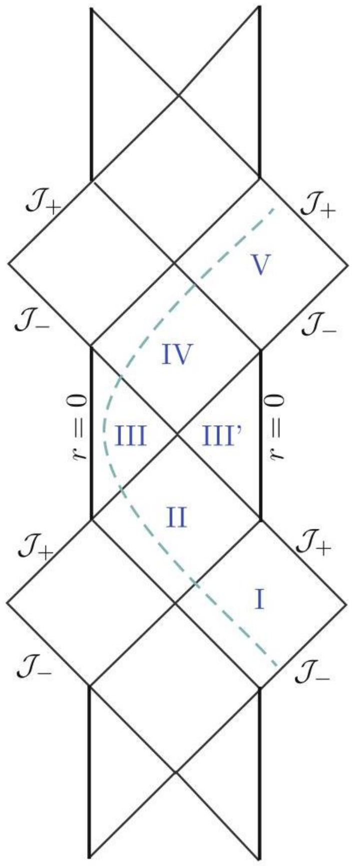

Fig. 4 Penrose diagram of the Bonanno-Reuter black hole (14)

图 4 博南诺-罗伊特黑洞的彭罗斯图 (14)

The dynamics of the whole process can be studied in more detail by exploiting a generalized Vaidya metric in Eddington-Finkelstein coordinates. Indeed, the evolution of the time-dependent mass $m(v)$ is dictated by the dynamical equation

我们可以利用爱丁顿-芬克斯坦坐标下的广义外迪亚度量，更细致地研究整个过程的动力学。实际上，含时质量 $m(v)$ 的演化由动力学方程决定

$$\dot{m}(v) = -L[m(v)] \tag{17}$$

where the black hole luminosity L (energy flux from the outer horizon of the black hole) is given by Stefan-Boltzmann law, $L = \sigma\left(4\pi r_+^2\right) T^4$. The result of the numerical integration for the classical and Bonanno-Reuter black holes is shown in Fig. 6. The important feature distinguishing Bonanno-Reuter (and, more generally, regular) black holes from their classical counterpart is the timing of the evaporation process: the final remnant configuration is only realized asymptotically, while the evaporation of Schwarzschild black holes occurs in a finite amount of time.

方程中黑洞光度 L (来自黑洞外视界的能流) 由斯特藩-玻尔兹曼定律给出，$L = \sigma\left(4\pi r_+^2\right) T^4$ 。经典黑洞与博南诺-罗伊特黑洞的数值积分结果如图 6 所示。博南诺-罗伊特黑洞 (更一般地说，正则黑洞) 区别于经典黑洞的重要特征是蒸发过程的时间特性: 最终残余构型仅能在渐近极限下实现，而史瓦西黑洞的蒸发会在有限时间内完成。

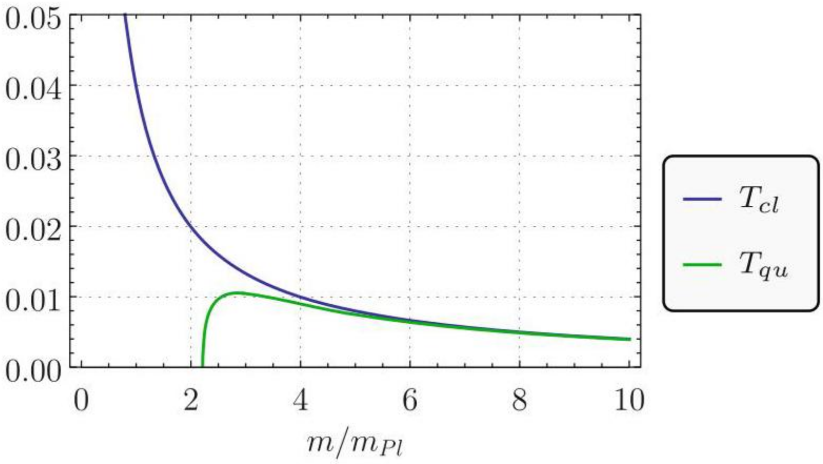

Fig. 5 Black hole temperature against the ADM mass m for classical (blue line) and RG-improved (green line) black holes. In the classical case $T \propto 1/m$, thus the black hole evaporates completely. As for Bonanno-Reuter black holes, the presence of two horizons causes the temperature to vanish at the critical mass $m_{cr} \sim m_{Pl}$. Accordingly, the evaporation process terminates after a finite amount of time and leaves behind a cold Planckian remnant

图 5 经典黑洞 (蓝线) 与 RG 改进黑洞 (绿线) 的黑洞温度随 ADM 质量 m 的变化。经典情况下 $T \propto 1/m$ ，因此黑洞会完全蒸发。对于博南诺-罗伊特黑洞，两个视界的存在使得温度在临界质量 $m_{cr} \sim m_{Pl}$ 处降为零。因此蒸发过程会在有限时间后终止，留下一个冷的普朗克残余

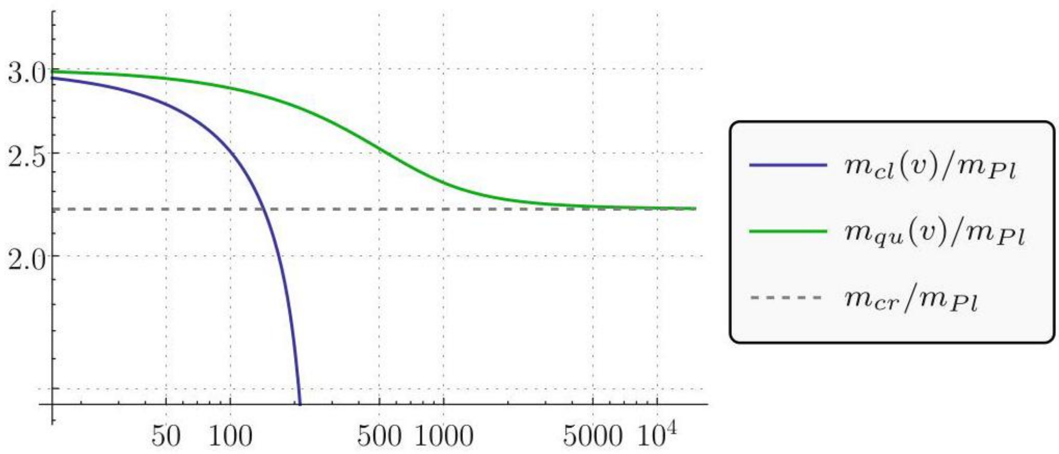

Fig. 6 Dynamics of the black hole ADM mass $m(v)$ in the classical (blue line) and RG-improved (green line) case. The evolution is shown via a log-log plot. For Schwarzschild black holes, the mass drops to zero in a finite amount of time, whereas Bonanno-Reuter black holes shrink to a finite size and only reach the final static configuration as $v \to \infty$

图 6 经典情况 (蓝线) 与 RG 改进情况 (绿线) 下黑洞 ADM 质量 $m(v)$ 的动力学，演化通过双对数图展示。史瓦西黑洞的质量会在有限时间内降至零，而博南诺-罗伊特黑洞会收缩至有限大小，仅在 $v \to \infty$ 时达到最终的静态构型

Thermodynamics of asymptotically safe black holes has been analyzed in great detail in [55]. Despite the different starting point, involving an RG improvement at the level of thermodynamical quantities (e.g., the black hole entropy, $S \to S_k = \frac{A}{4G_k}$) and the use of a different,"optimized" cutoff $k = k_{\text{opt}}(m, J, Q) \propto 1/A \sim 1/r_s^2$ which can in principle depend on the angular momentum J and charge Q of the black hole, the results are in qualitative agreement with those presented above.

渐近安全黑洞的热力学已经在文献 [55] 中得到了非常细致的分析。尽管该工作的出发点不同——它对热力学量 (例如黑洞熵 $S \to S_k = \frac{A}{4G_k}$) 层面做 RG 改进，并且使用了不同的“优化”截断 $k = k_{\text{opt}}(m, J, Q) \propto 1/A \sim 1/r_s^2$ ，原则上该截断可以依赖黑洞的角动量 J 和电荷 Q ，但得到的结果和上文所述在定性上一致。

Finally, it is worth mentioning that Bonanno-Reuter black holes have been successfully used in the literature to investigate potential implications of quantum gravity. Particularly, quasi-normal modes of Bonanno-Reuter black holes have been studied in [72], and it was shown that for Planckian black holes quasi-normal modes display important deviations from the classical case.

最后，值得一提的是，博南诺-罗伊特黑洞已经在文献中被成功用于研究量子引力的潜在影响。特别地，文献 [72] 研究了博南诺-罗伊特黑洞的准正则模式，结果表明，对于普朗克黑洞，其准正则模式和经典情况存在显著偏差。

After the pioneering work of Bonanno and Reuter [48-50], several extensions and applications within the spherically symmetric case have been considered. In the following, we briefly summarize modifications induced by the inclusion of extra dimensions.

在博南诺与罗伊特的开创性工作 [48-50] 之后，学界已经在球对称情形下完成了若干扩展与应用。下文我们简要总结引入额外维度带来的修正。

Extra Dimensions [51,52]

额外维度 [51,52]

The derivation of Bonanno-Reuter black holes relies on the physical input that the number of large dimensions is $d = 4$. Their generalization to $d \geq 4$ dimensions is relevant in quantum gravity models with large extra dimensions [73]; this was the focus of the studies in [51,52]. If our universe were d -dimensional, the Planck mass $m_{Pl}^2 = V_{d-4} m_d^{d-2}$ would be affected by the compactified volume of extra dimensions, V_{d-4} , and the lapse function of Schwarzschild-like black holes would read

博南诺-罗伊特黑洞的推导依赖于大维度数量为 $d = 4$ 这一物理输入。将其推广到 $d \geq 4$ 维，对含大额外维度的量子引力模型 [73] 而言十分重要；这正是文献 [51,52] 的研究核心。若我们的宇宙是 d 维的，普朗克质量 $m_{Pl}^2 = V_{d-4} m_d^{d-2}$ 会受额外维度紧致体积 V_{d-4} 的影响，类史瓦西黑洞的 lapse 函数形式为

$$f_{cl}(d,r) = 1 - \frac{2mG_0}{r^{d-3}}. \tag{18}$$

The d -dimensional Schwarzschild radius, $r_s(d) = \sqrt[d-3]{2mG_0}$, would thus be smaller than in general relativity. Interestingly, in the large radii limit, the integrand defining the proper distance $D(r)$ is independent of d and thus $D(r) \sim r$ asymptotically for any spacetime dimension. The d -dimensional analog of Eq. (13) is

因此，d 维史瓦西半径 $r_s(d) = \sqrt[d-3]{2mG_0}$ 会小于广义相对论中的结果。有意思的是，在大半径极限下，定义固有距离 $D(r)$ 的被积函数与 d 无关，因此对任意时空维数，$D(r) \sim r$ 都是渐近成立的。式 (13) 的 d 维类比形式为

$$D(r) \approx \frac{2r^{\frac{d-1}{2}}}{(d-1)} \left(r_s + \left(\frac{d-1}{2}\right)^{-\frac{2}{d-3}} r \right)^{\frac{3-d}{2}} \tag{19}$$

and, following the procedure to derive Bonanno-Reuter black holes, one can use this analytic approximation to set the scale, $k \sim 1/D(r)$. This yields a metric with lapse function

并且，遵循推导博南诺-罗伊特黑洞的步骤，你可以用这个解析近似来设定标度 $k \sim 1/D(r)$ ，由此得到带如下 lapse 函数的度规

$$f_{qu}(d,r) = 1 - \frac{2mG_0}{r^{d-3}} \frac{r^{\frac{(d-1)(d-2)}{2}}}{r^{\frac{(d-1)(d-2)}{2}} + g_*^{-1} G_0 \left(\frac{d-1}{2}\right)^{d-2} \left(r_s + \left(\frac{d-1}{2}\right)^{-\frac{2}{d-3}} r \right)^{\frac{d^2+6-5d}{2}}}.$$

(20)

The horizon structure and thermodynamical properties of this class of black holes in higher dimensions are analogous to those of Bonanno-Reuter black holes, with two horizons and similar dynamical evaporation.

这类高维黑洞的视界结构与热力学性质和博南诺-罗伊特黑洞类似，都存在两个视界，且动力学蒸发过程也相近。

While the existence of extra dimensions remains unproven, and their theoretical impact of limited use, the presence of a positive cosmological constant is currently strongly supported by observations. Its role in the context of asymptotically safe black holes is the focus of the next subsection.

尽管额外维度的存在仍未得到证实，其理论影响也十分有限，但正宇宙常数的存在目前得到了观测的强力支持。它在渐近安全黑洞框架下的作用是下一小节的讨论核心。

The Role of the Cosmological Constant

宇宙学常数的作用

Asymptotically flat RG-improved black holes have been extensively studied in the literature, with the result that gravitational antiscreening yields at least a weakening of the classical singularity. In this subsection we discuss how singularity resolution may be affected by the introduction of a cosmological constant [74, 75] and by higher-derivative terms in the action [76].

渐近平坦 RG 改进黑洞已在文献中被广泛研究，结果表明引力反屏蔽至少会弱化经典奇点。在本小节中，我们讨论引入宇宙学常数 [74, 75] 和作用量中的高阶导数项会如何影响奇点解析 [76]。

Conditions for Singularity Resolution and the Cosmological Constant [74,75] In the case of asymptotically de Sitter spacetimes, the modified mass function reads

奇点解析条件与宇宙学常数 [74,75] 在渐近德西特时空的情况下，修正质量函数可写为

$$f_{qu}(r) = 1 - \frac{2mG_k}{r} - \frac{1}{3}\Lambda_k r^2, \tag{21}$$

where G_k and Λ_k are the dimensionful running Newton and cosmological couplings and $k = k(r)$. This class of RG-improved black holes was first considered in [74], where the authors argued that the introduction of a cosmological constant $\Lambda_k \neq 0$ could be problematic in non-unimodular settings. The argument goes as follows. In an asymptotically safe regime, in $d = 4$, the running Newton and cosmological couplings scale as

其中 G_k 和 Λ_k 是有量纲的跑动牛顿耦合与宇宙学耦合，且 $k = k(r)$ 。这类 RG 改进黑洞最早在文献 [74] 中被研究，作者提出，在非幺模框架下引入宇宙学常数 $\Lambda_k \neq 0$ 可能存在问题。论证如下: 在渐近安全区域，$d = 4$ 中，跑动牛顿耦合与宇宙学耦合标度为

$$G_k \sim g_* k^{-2},\ \Lambda_k \sim \lambda_* k^2. \tag{22}$$

The momentum k is then to be related to physical quantities, such as curvature invariants. On dimensional grounds and due to spherical symmetry, the functional relation $k(r)$ is universal in the UV (modulo a numerical factor of $O(1)$)

动量 k 接下来会与曲率不变量这类物理量关联。从量纲分析出发，且由于球对称性，函数关系 $k(r)$ 在紫外区是普适的 (除 $O(1)$ 的数值因子外)

$$k \sim D(r)^{-1} \propto K(r)^{1/4} \propto r^{-3/2}, \tag{23}$$

where $K(r)$ is the Kretschmann scalar. Consequently, close to the classical singularity, any scale identification maps the classical Schwarzschild-de Sitter black hole with Newton coupling G_0 and cosmological constant Λ_0 into another Schwarzschild-de Sitter solution with Newton coupling $\widetilde{G} \simeq 3\lambda_*/4G_0$ and cosmological constant $\widetilde{\Lambda} \simeq 4g_*/(3G_0)$. As a result, unless $\lambda_* \neq 0$, the inclusion of a cosmological constant Λ_0 seems both to reintroduce the singularity and also to restore the thermodynamical properties of classical Schwarzschild-de Sitter black holes [4, 56] .[4]

其中 $K(r)$ 是克雷奇曼标量。因此，在经典奇点附近，任何标度识别都会将带有牛顿耦合 G_0 和宇宙学常数 Λ_0 的经典史瓦西-德西特黑洞映射为另一个带有牛顿耦合 $\widetilde{G} \simeq 3\lambda_*/4G_0$ 和宇宙学常数 $\widetilde{\Lambda} \simeq 4g_*/(3G_0)$ 的史瓦西-德西特解。结果是，除非 $\lambda_* \neq 0$，引入宇宙学常数 Λ_0 似乎既会重新引入奇点，也会恢复经典史瓦西-德西特黑洞的热力学性质 [4, 56] .[4]

Since a UV-vanishing cosmological constant would restore the regularity of RG-improved black holes, a key question is whether $\lambda_* = 0$ is compatible with the spacetime being asymptotically de Sitter, i.e., with an ”emergent” positive cosmological constant. This question has to do with the conditions for black hole singularity resolution analyzed in [75].

既然紫外区趋于零的宇宙学常数可以恢复 RG 改进黑洞的正则性，核心问题在于 $\lambda_* = 0$ 是否与时空渐近德西特 (即存在“涌现”正宇宙学常数) 相容。该问题与文献 [75] 中分析的黑洞奇点解析条件直接相关。

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[4] In a unimodular approach to quantum gravity, the cosmological constant emerges as an integration constant instead of a coupling, and thus it does not reintroduce the Schwarzschild singularity [77].

[4] 在量子引力的幺模方法中，宇宙学常数是作为积分常数而非耦合涌现的，因此不会重新引入史瓦西奇点 [77]。

On dimensional grounds, the leading-order scaling of the IR momentum cutoff $k(r)$ close to the classical singularity of a spherically symmetric black hole is

从量纲分析出发，球对称黑洞经典奇点附近红外动量截断 $k(r)$ 的领头阶标度为

$$k(r) = \xi (mG_0)^{\gamma-1} r^{-\gamma}, \tag{24}$$

where $\gamma > 0$ by consistency. Although in the spherically symmetric case dimensional analysis and physical consideration imply $\gamma = 3/2$ (cf. Eq. (23)) in the following we will keep it unspecified.

其中自洽性要求满足 $\gamma > 0$。尽管在球对称情形下，量纲分析和物理考虑都暗示 $\gamma = 3/2$ (参见式 (23))，在下文中我们仍不将其取定。

In order to determine the conditions for black hole singularity resolution, it is convenient to rewrite the lapse function as

为了确定黑洞奇点解析的条件，将迟滞函数改写为如下形式更方便

$$f_{qu}(r) = 1 - 2\Phi(r) = 1 - 2\Phi_G(r) - 2\Phi_\Lambda(r), \tag{25}$$

i.e., in terms of the ”pseudo Newtonian potential” $\Phi(r)$ (see also section ”Toward a Dressed Newtonian Potential Within Asymptotic Safety”) and its components

也就是用“伪牛顿势”$\Phi(r)$ (另见“渐近安全框架下的修整牛顿势”一节) 及其分量表示

$$\Phi_G(r) = \frac{mG(r)}{r},\ \Phi_\Lambda(r) = \frac{\Lambda(r)}{6}r^2, \tag{26}$$

with $G(r) = G[k(r)]$ and $\Lambda(r) = \Lambda[k(r)]$. The corresponding Ricci and Kretschmann scalars

满足 $G(r) = G[k(r)]$ 和 $\Lambda(r) = \Lambda[k(r)]$ 。对应的里奇标量与克雷奇曼标量

$$R = \frac{4\Phi}{r^2} + \frac{8\Phi'}{r} + 2\Phi'',\ K = \frac{16\Phi^2}{r^4} + \frac{16\Phi'^2}{r^2} + 4\Phi''^2, \tag{27}$$

receive contribution from three quantities: $\Phi/r^2, \Phi'/r$, and Φ'' . The two invariants are thereby regular at $r = 0$ if the leading-order scaling of $\Phi(r)$ close to the classical singularity is $\Phi(r) \sim r^\delta$ with $\delta \geq 2$. This condition can in turn be translated into constraints on γ and on the RG flow of the dimensionless Newton coupling $g_k = G_k k^2$ and cosmological constant $\lambda_k = \Lambda_k k^{-2}$ [75].

由三个量提供贡献: $\Phi/r^2, \Phi'/r$ 和 Φ'' 。若经典奇点附近 $\Phi(r)$ 的领头阶标度为满足 $\delta \geq 2$ 的 $\Phi(r) \sim r^\delta$ ，则这两个不变量在 $r = 0$ 处正则。该条件可进一步转化为对 γ ，以及对无量纲牛顿耦合 $g_k = G_k k^2$ 和宇宙学常数 $\lambda_k = \Lambda_k k^{-2}$ 的 RG 流的约束 [75]。

The behavior of g_k and λ_k in the proximity of the fixed-point regime (22) is obtained by linearizing their beta functions about the fixed point. For a general set of couplings $\{g_i(k)\}$, the linearized beta functions read

对不动点区域 (22) 附近 g_k 和 λ_k 的行为，可以通过将它们的 β 函数在不动点处线性化得到。对于一般的耦合集合 $\{g_i(k)\}$ ，线性化后的 β 函数为

$$k\partial_k g^i(k) = \sum_{j=1}^{n} \left.\frac{\partial\beta^i}{\partial g^j}\right|_{\mathbf{g}_*} \left(g^j(k) - g^j_*\right) + O\left(g^j(k) - g^j_*\right)^2, \tag{28}$$

and, to linear order, the solution takes the form

且在线性阶下，解的形式为

$$g^i(k) = g^i_* + \sum_{j=1}^{n} c^i_j V^i_j \left(\frac{k}{M_\mathrm{P}}\right)^{-\theta_j}. \tag{29}$$

In this expression c^j_i are integration constants selecting the specific RG trajectories, θ_i are critical exponents, and V^i_j are the components of the corresponding eigendi-rections. Positive critical exponents thus correspond to IR relevant directions, and, vice versa, negative critical exponents identify the irrelevant ones. Similarly to the case of perturbative gauge or matter theories, where one drops out the irrelevant operators in order to select the subset of RG trajectories that are asymptotically free, here we can select the integration constants in front of irrelevant deformations to zero; this choice selects the RG trajectories belonging to the basin of attraction of the fixed point $\{g^i_*\}$. Yet, at variance of perturbative theories, since g^i_* is generally non-zero, selecting the UV critical surface of the fixed point is not equivalent to discarding operators associated with irrelevant directions in the (bare) action.

在该表达式中，c_i^j 是选定特定 RG 轨迹的积分常数，θ_i 是临界指数，V_j^i 是对应本征方向的分量。因此正临界指数对应红外相关方向，反之负临界指数对应非相关方向。与微扰规范理论或物质理论类似——在这些理论中人们会舍弃非相关算符，以选出渐近自由的 RG 轨迹子集，在此我们也可以将非相关变形前的积分常数设为零；该选择选出了属于不动点 $\{g_*^i\}$ 吸引域的 RG 轨迹。但与微扰理论不同的是，由于 g_*^i 通常非零，选定不动点的紫外临界面不等同于丢弃 (裸) 作用量中非相关方向对应的算符。

Restricting ourselves to the Einstein-Hilbert subspace, which is the object of the investigations in [75], the scaling of g_k and λ_k close to the fixed point reads

当我们限制在爱因斯坦-希尔伯特子空间 (即文献 [75] 的研究对象) 时，不动点附近 g_k 和 λ_k 的标度为

$$g_k = g_* + g_1\left(\frac{k}{M_\mathrm{P}}\right)^{-\theta_1} + g_2\left(\frac{k}{M_\mathrm{P}}\right)^{-\theta_2}, \tag{30a}$$

$$\lambda_k = \lambda_* + \lambda_1\left(\frac{k}{M_\mathrm{P}}\right)^{-\theta_1} + \lambda_2\left(\frac{k}{M_\mathrm{P}}\right)^{-\theta_2}, \tag{30b}$$

and is crucially determined by the critical exponents θ_i . Using these expressions, one can easily show that Φ_G gives a non-singular contribution to the curvature invariants for $\gamma \geq 3/2$ [75]. Moreover, the requirement of singularity resolution yield constraints on the critical exponents θ_i [75].

且该标度由临界指数 θ_i 决定性地确定。利用这些表达式可以很容易证明，对 $\gamma \geq 3/2$ 而言 Φ_G 对曲率不变量给出非奇异贡献 [75]。此外，奇点消解的要求也会对临界指数 θ_i 给出约束 [75]。

The first constraint arises from the form of Φ_G : the critical exponents are required to be positive. This is intuitive, since singularity resolution is associated with an effective weakening of the gravitational interaction at high energies. This weakening can come from a gravitational antiscreening only if the theory is asymptotically safe, i.e., if the critical exponents associated with G and Λ are positive.

第一个约束来自 Φ_G 的形式: 要求临界指数为正。这很直观，因为奇点消解与高能下引力相互作用的有效弱化相关。只有当理论渐近安全，即与 G 和 Λ 关联的临界指数为正时，这种弱化才能通过引力反屏蔽实现。

Additional constraints come from Φ_Λ : a UV-attractive non-trivial fixed point of the gravitational RG flow implies a UV-scaling for the cosmological constant of the form $\Lambda_k \sim \lambda_* k^2$. Although geodesic completeness of an RG-improved Schwarzschild-(A)dS black hole requires λ_* to be zero [74], if one assumes the spacetime to be asymptotically de Sitter, a positive cosmological constant ought to re-emerge dynamically. Such a re-emergence is possible in principle, since the corrections to the fixed-point scaling (22) become important away from the fixed point and can drive Λ_k to a non-zero value. Not all critical exponents would however be compatible with both singularity resolution and the emergence of a positive cosmological constant. Indeed, these physical conditions imply that the critical exponents should satisfy the inequality $\theta_i \geq 2$. In turn, this bound is in agreement with many FRG computations, e.g., [18, 78, 79], and ensures that the cosmological constant could vanish fast enough as $r \to 0$ as not to reintroduce a curvature singularity.

额外约束来自 Φ_Λ : 引力 RG 流的紫外吸引非平凡不动点意味着宇宙常数具有 $\Lambda_k \sim \lambda_* k^2$ 形式的紫外标度关系。尽管 RG 改进的施瓦西-(A)dS 黑洞的测地完备性要求 λ_* 为零 [74]，但如果假设时空渐近于德西特空间，正宇宙常数应当会动力学重新出现。这种重新出现原则上是可能的，因为远离不动点时，对不动点标度关系 (22) 的修正会变得重要，并能驱动 Λ_k 变为非零值。但并非所有临界指数都能同时满足奇点消解和正宇宙常数出现的要求。事实上，这些物理条件要求临界指数满足不等式 $\theta_i \geq 2$ 。该界限与许多 FRG 计算结果一致，例如文献 [18, 78, 79]，并且它保证了宇宙常数随 $r \to 0$ 衰减得足够快，不会重新引入曲率奇点。

Impact of Higher Derivatives [76]

高阶导数的影响 [76]

Given the general form of the effective action (3), checking the stability of the results obtained within the Einstein-Hilbert truncation against the introduction of higher-derivative terms is of crucial importance. As a first step, Cai and Easson considered an effective action of the form (2) with constant form factors $g_{i,k}(\Box) \equiv g_{i,k}$ and truncated it to quadratic order [76]. In such a truncation the class of allowed spacetimes is in principle much richer [80], but still includes Schwarzschild-like solutions. To determine the impact of higher derivatives on Bonanno-Reuter black holes, [76] imposed the starting metric to be of the Schwarzschild-de Sitter form. The scale-dependent lapse function is then given by Eq. (21), and the RG improvement involves two couplings as before. However, at variance of the analysis in [48 − 50, 74, 75] , where on dimensional grounds $k(r) \sim r^{-3/2}$ (cf. Eq. (23)), the presence of higher-derivatives introduces new mass scales that can in principle modify Eq. (23) or at least render it non-trivial. Specifically, the cutoff function can be determined by the condition that the trace of the generalized Einstein tensor $\widetilde{G}_{\mu\nu}$ (which includes contributions from higher derivatives up to quadratic order) be zero, i.e., the modified field equations to be satisfied [76]

鉴于有效作用量 (3) 的一般形式，检验爱因斯坦-希尔伯特截断所得结果对引入高阶导数项的稳定性至关重要。作为第一步，蔡和伊森考虑了具有常数形状因子 $g_{i,k}(\Box) \equiv g_{i,k}$ 的形式为 (2) 的有效作用量，并将其截断至二阶 [76]。在该截断下，允许时空的种类原则上丰富得多 [80]，但仍包含类史瓦西解。为了确定高阶导数对博南诺-罗伊特黑洞的影响，文献 [76] 要求初始度规满足史瓦西-德西特形式。依赖尺度的时移函数由式 (21) 给出，重整化群改进和之前一样涉及两个耦合常数。但与 [48 − 50, 74, 75] 中的分析不同，在量纲分析下满足 $k(r) \sim r^{-3/2}$ (参见式 (23))，高阶导数的存在引入了新的质量标度，原则上可以修改式 (23)，或至少使其不再平凡。具体而言，截断函数可由广义爱因斯坦张量 $\widetilde{G}_{\mu\nu}$ (包含最高到二阶高阶导数的贡献) 的迹为零即修正场方程成立这一条件确定 [76]

$$\mathrm{Tr}\left(\widetilde{G}_{\mu\nu}\right) = \frac{k^2}{8\pi g_k}\left(4\lambda_k k^2 - R\right) - 6g_R \Box R = 0. \tag{31}$$

This strategy is along the lines of the cutoff identification by Bianchi identities that was formally devised and applied in [43, 46, 47, 81, 82] and that we will discuss in more detail in section ”Constraints on the Cutoff Identification from Bianchi Identities.” This procedure is more rigorous than the one employed in [48-50]. On the other hand, this strategy is not applicable in the pure Einstein-Hilbert truncation, in which the above relation becomes trivial.

该方案遵循了由比安基恒等式确定截断的思路，这一思路已在 [43, 46, 47, 81, 82] 中正式提出并应用，我们会在“比安基恒等式对截断确定的约束”一节中更详细地讨论。该过程比文献 [48-50] 中使用的方法更严谨。另一方面，该方案不适用于纯爱因斯坦-希尔伯特截断，在纯截断下上述关系会变得平凡。

Plugging the expression of the curvature invariants and running couplings in Eq. (31), one finds $k \propto r^{-3/4}$ [76], leading to a black hole whose central singularity is weaker than in the classical case albeit not fully resolved. Other than the nature of the singularity, Cai-Easson black holes share all properties of Bonanno-Reuter black holes, from the number of horizons to their thermodynamics. At the same time, the different scaling enforced by the validity of Bianchi identities in the presence of higher-derivatives removes the issues induced by a cosmological constant in the resolution of black hole singularities in the Einstein-Hilbert truncation.

将曲率不变量和跑动耦合的表达式代入式 (31)，可得 $k \propto r^{-3/4}$ [76]，由此得到的黑洞中心奇点比经典情况更弱，但并未完全解决。除奇点性质外，蔡-伊森黑洞与博南诺-罗伊特黑洞的所有性质都相同，从视界数量到热力学性质都一致。同时，在存在高阶导数的情况下，由比安基恒等式成立所要求的不同标度行为，消除了爱因斯坦-希尔伯特截断中解决黑洞奇点问题时由宇宙常数引发的问题。

Having discussed static and spherically symmetric configurations, as well as the role of the cosmological constant, we are now ready to add another key ingredient into the game: rotation.

讨论完静态球对称构型以及宇宙常数的作用后，我们现在可以引入另一个关键要素: 旋转。

Rotating RG-Improved Black Holes and Their Shadows

旋转型 RG 改进黑洞及其阴影

This subsection discusses RG-improved black holes with non-vanishing angular momentum and their shadows [83-85] (see also [10]). Some variations on the topic, mostly based on different scale identifications, can be found in [86,87].

本小节讨论具有非零角动量的 RG 改进黑洞及其阴影 [83-85](另见 [10])。该主题的部分变体大多基于不同的标度识别，可在 [86,87] 中找到相关内容。

Reuter-Tuiran Rotating Black Holes [83,84]

Reuter-Tuiran 旋转黑洞 [83,84]

We start by reviewing one (and the first) derivation of the RG-improved metric of a rotating black hole. Spinning black holes are classically described by the Kerr metric. In Boyer-Lindquist coordinates, it reads

我们首先回顾旋转黑洞 RG 改进度规的首个推导之一。经典层面上，旋转黑洞由克尔度规描述。在博耶-林德奎斯特坐标下，其形式为

$$
\begin{aligned}
ds^2 = & -\frac{\Delta - a^2\sin^2\theta}{\rho^2}dt^2 + \frac{\rho^2}{\Delta}dr^2 \\
& + \frac{\left(a^2 + r^2\right)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta d\phi^2 \\
& + \rho^2 d\theta^2 - \frac{2\left(a^2 + r^2 - \Delta\right)}{\rho^2}a\sin^2\theta dt d\phi.
\end{aligned}
\tag{32}
$$

where $a \equiv J/m \in \left[0, m/m_{Pl}^2\right]$ is the specific angular momentum and

其中 $a \equiv J/m \in \left[0, m/m_{Pl}^2\right]$ 为比角动量，且

$$
\rho^2 = r^2 + a^2\cos^2\theta,\ \Delta = r^2 + a^2 - 2G_0 mr. \tag{33}
$$

Kerr metrics constitute a two-parameter family of solutions, the two parameters being the black hole mass m and its angular momentum J . A non-zero angular momentum also implies that spherical symmetry turns into axial symmetry with respect to the rotation axis.

克尔度规构成一个双参数解族，两个参数分别为黑洞质量 m 和角动量 J 。非零角动量还意味着球对称转变为相对于自转轴的轴对称。

In [83, 84] the authors followed the Bonanno-Reuter procedure closely and exploited the proper distance to set the scale. The proper distance from a given spacetime point to the origin along a radial geodesic is now a function of θ :

在文献 [83,84] 中，作者严格遵循 Bonanno-Reuter 方法，利用固有距离设定标度。从给定时空点到原点沿径向测地线的固有距离是关于 θ 的函数:

$$
D(r, \theta) = \int_0^r d\bar{r}\sqrt{\left|\frac{\bar{r}^2 + a^2\cos^2\theta}{\bar{r}^2 + a^2 - 2mG_0\bar{r}}\right|}. \tag{34}
$$

This function can generally be obtained numerically, while an analytic expression is only possible by restricting θ , e.g., to the equatorial plane $\theta = \pi/2$. As the θ - dependence is rather weak, [83,84] neglected it to derive the qualitative features of the resulting spacetime. This can be done, for instance, by setting $\theta = \pi/2$.

该函数一般可通过数值求解得到，只有对 θ 加以限制 (例如限制在赤道面 $\theta = \pi/2$) 才能得到解析表达式。由于对 θ 的依赖相当微弱，文献 [83,84] 忽略了这种依赖来推导所得时空的定性特征，例如可以通过令 $\theta = \pi/2$ 实现这一点。

For sufficiently large masses (away from the extremality condition), the spacetime resulting from replacing $G_0 \to G\left[d(r)^{-1}\right]$ in the classical Kerr metric is characterized by two infinite redshift surfaces located at $r_{S_\pm}(\theta)$ such that $g_{tt} = 0$, i.e.,

对于足够大的质量 (远离极端性条件)，在经典克尔度规中替换 $G_0 \to G\left[d(r)^{-1}\right]$ 后得到的时空具有两个无限红移面，分别位于 $r_{S_\pm}(\theta)$，满足 $g_{tt} = 0$，即

$$r^2 - 2G(r)\,mr + a^2\cos^2\theta = 0, \tag{35}$$

the outer one being the static limit surface. In addition, the Reuter-Tuiran spacetime features two horizons whose radii $r_\pm$ are the solutions to the equation $g_{rr} = 0$, i.e.,

外侧的那个是静限界。此外，Reuter-Tuiran 时空具有两个视界，其半径 $r_\pm$ 是方程 $g_{rr} = 0$ 的解，即

$$r^2 - 2G(r)\,mr + a^2 = 0. \tag{36}$$

These values are to be found numerically. Similarly to the case of spherically symmetric spacetimes, the two horizons coalesce when the black hole mass m is decreased to a threshold value of the order of the Planck mass. In this process, also the two infinite redshift surfaces merge and then disappear.

这些数值需要通过数值方法求解。与球对称时空的情况类似，当黑洞质量 m 减小到普朗克质量量级的阈值时，两个视界会合并；在此过程中，两个无限红移面也会随之合并而后消失。

The Reuter-Tuiran black hole is not a solution to the vacuum Einstein equations, but rather to modified Einstein equations of the form

Reuter-Tuiran 黑洞并不是真空爱因斯坦方程的解，而是如下形式的修正爱因斯坦方程的解

$$G_{\mu\nu} = 8\pi G_0 T^{\text{eff}}_{\mu\nu}, \tag{37}$$

with an effective energy-momentum tensor $T^{\text{eff}}_{\mu\nu}$. As a consequence, the bare black hole mass m and angular momentum J are replaced by corresponding RG-improved quantities, whose value can be computed via the Komar integrals

其中包含有效能量动量张量 $T^{\text{eff}}_{\mu\nu}$ 。因此，裸黑洞质量 m 和角动量 J 被替换为对应的 RG 改进量，其数值可以通过科马尔积分计算得到

$$M^\infty_{qu} = -(8\pi G_0)\oint \nabla^\alpha t^\beta dS_{\alpha\beta}, \tag{38a}$$

$$J^\infty_{qu} = (16\pi G_0)\oint \nabla^\alpha \phi^\beta dS_{\alpha\beta}, \tag{38b}$$

In the expressions above, t_β and ϕ_β are the Killing vectors associated with the invariance of the system under time translations and rotations around the spinning axis, respectively. Moreover, the integration is over a two-sphere S at spatial infinity, so that $S_{\alpha\beta} = -2n_{[\alpha}r_{\beta]}\sqrt{\sigma}d^2\theta$, where n_α and r_α are timelike and spacelike normal vectors to S, σ is the determinant of the spatial metric $\sigma_{\alpha\beta}$ induced by $g_{\mu\nu}$ on S , and $d^2\theta = d\theta_1 d\theta_2$ with θ_i being angular coordinates on S . Since the RG-improved metric reproduces the classical one asymptotically, the Komar integrals above coincide with their classical counterparts. In contrast, if the two-sphere S is the one enclosed by the outer horizon r_+ , the modified Komar integrals [83, 84]

在上述表达式中，t_β 和 ϕ_β 分别是系统在时间平移和绕自转轴旋转下的不变性对应的基林矢量。此外，积分是对空间无穷远处的二维球面 S 进行的，因此有 $S_{\alpha\beta} = -2n_{[\alpha}r_{\beta]}\sqrt{\sigma}d^2\theta$，其中 n_α 和 r_α 分别是 S,σ 的类时和类空法向量，$\sigma_{\alpha\beta}$ 是 $g_{\mu\nu}$ 在 S 上诱导的空间度规的行列式，$d^2\theta = d\theta_1 d\theta_2$，其中 θ_i 是 S 上的角坐标。由于 RG 改进度规在渐近区域还原为经典度规，上述科马尔积分与经典形式一致。反之，如果二维球面 S 是被外视界 r_+ 包围的球面，修正后的科马尔积分 [83, 84]

$$M_{qu} = \frac{mG(r_+)}{G_0}\left[1 - \arctan\left(\frac{a}{r_+}\right)\frac{G'(r_+)\left(r_+^2 + a^2\right)}{aG(r_+)}\right], \tag{39a}$$

$$J_{qu} = \frac{JG(r_+)}{G_0} + \frac{M^2 r_+^2 G'(r_+)\,G(r_+)}{G_0 a}\left[1 - \frac{2MG(r_+)}{a}\arctan\left(\frac{a}{r_+}\right)\right],$$

(39b)

only reduce to the bare values in the limit $G(r) \to G_0$. Independent of the specific form of $G(r)$, the effective mass M_{qu} is smaller than its classical counterpart, as expected from the gravitational antiscreening. Yet, Smarr's relation between the mass and angular momentum

仅在极限 $G(r) \to G_0$ 下退 bare 裸值。无论 $G(r)$ 的具体形式如何，有效质量 M_{qu} 都小于经典对应值，这与引力反屏蔽的预期一致。不过，质量与角动量之间的斯马尔关系

$$M_{qu} = 2\Omega_{qu}J_{qu} + (4\pi G_0)^{-1}\kappa_{qu}A_{qu}, \tag{40}$$

with the improved surface gravity, area, and angular frequency given by

改进后的表面引力、面积和角频率由下式给出

$$\kappa_{qu} = (r_+ - 2mG_0)\left(G(r_+) + r_+G'(r_+)\right)\left(r_+^2 + a^2\right)^{-1}, \tag{41a}$$

$$A_{qu} = 4\pi\left(r_+^2 + a^2\right),\ \Omega_{qu} = a\left(r_+^2 + a^2\right)^{-1}, \tag{41b}$$

still holds in the same form as in the classical case.

仍保持与经典情况相同的形式成立。

Black Hole Shadows [85]

黑洞阴影 [85]

In [85] the authors considered a class of RG-improved spinning black holes and studied the size and shape of the corresponding black hole shadow. The cutoff identification is set by $k^4 \simeq K, K$ being the Kretschmann scalar. For a static spacetime, $K = 48G_0^2m^2/r^6$, and the RG improvement maps the classical Schwarzschild spacetime into the Hayward metric [88]. In the case of rotating black holes, the Kretschmann scalar also depends on the angular momentum a and the angular coordinate θ :

文献 [85] 中，作者研究了一类 RG 改进旋转黑洞，并分析了对应黑洞阴影的大小与形状。截断标识由 $k^4 \simeq K, K$ 给出，其中 $k^4 \simeq K, K$ 是克雷奇曼标量。对于静态时空，$K = 48G_0^2m^2/r^6$，RG 改进将经典史瓦西时空映射为海沃德度规 [88]。在旋转黑洞的情况下，克雷奇曼标量还依赖于角动量 a 和角坐标 θ:

$$K(r,\theta,a) = \frac{48G_0^2m^2}{\left(r^2+a^2\cos(\theta)^2\right)^6}\left(r^6 - 15r^4a^2\cos(\theta)^2 + 15r^2a^4\cos(\theta)^4 - a^6\cos(\theta)^6\right). \tag{42}$$

The authors of [85] have been the first to employ a θ -dependent cutoff identification to RG-improved Kerr black holes. However, since the polynomial

文献 [85] 的作者首次将依赖 θ 的截断标识应用于 RG 改进克尔黑洞。然而，由于该多项式

$$-15r^4a^2\cos(\theta)^2 + 15r^2a^4\cos(\theta)^4 - a^6\cos(\theta)^6 \tag{43}$$

is not everywhere positive, it could lead to negative k^2 and, consequently, to a complex-valued spacetime metric. To avoid this issue, [85] neglected this part of the Kretschmann scalar and carried out the RG improvement by replacing $G_0 \to G_k$ in Eq. (32), with $k^2 \equiv G_0mr^3/\left(r^2+a^2\cos(\theta)^2\right)^3$ (see [89,90] for generalizations which exploit horizon-penetrating coordinates and do not neglect the angular dependence). As for the spherically symmetric case, the event horizon is located inside the classical Schwarzschild radius, resulting in a smaller black hole (and thus in a more compact shadow). Moreover, the higher is the specific angular momentum a , the stronger the dependence on θ is. At this point, one can determine the shadow by looking at how light rays are deviated by a black hole. Concretely, this is done by solving their geodesic equation backward, from a localized distant observer (the camera) to a source.

并非处处为正，它可能导致 k^2 为负，进而得到复值的时空度规。为了解决这个问题，文献 [85] 忽略了克雷奇曼标量的这部分分量，通过将式 (32) 中的 $G_0 \to G_k$ 替换为 $k^2 \equiv G_0mr^3/\left(r^2+a^2\cos(\theta)^2\right)^3$ 完成 RG 改进 (相关推广见文献 [89,90]，这些推广利用穿视界坐标，没有忽略角度依赖)。和球对称情况一样，事件视界位于经典史瓦西半径内侧，形成的黑洞更小 (因此阴影也更紧致)。此外，比角动量 a 越大，对 θ 的依赖就越强。在此基础上，我们可以通过分析光线被黑洞偏折的方式确定阴影形状，具体做法是从局域遥远观测者 (相机) 反向追溯求解光线的测地线方程，直到光源。

The deviation of the resulting shadow from the one expected from a singular, classical rotating black hole depends strongly on the scale at which quantum gravity effects become important. Enforcing quantum gravity effects at large scales clearly magnifies any deviation. The two major deviations consist of:

所得阴影与经典奇异旋转黑洞预期阴影的偏差，强烈依赖于量子引力效应发挥作用的能标。如果量子引力效应在大尺度上显现，偏差会明显放大。两种主要偏差为:

- A shrinking of the shadow, due to an event horizon that is smaller than its classical counterpart. This effect is at work for both the spherically and the axially symmetric cases.

- 阴影缩小，原因是事件视界比经典对应物更小。该效应在球对称和轴对称情形下均存在。

- In the case of spinning spacetimes, the appearance of a characteristic ”dent” [85,89-95]. [5] The latter is more evident on the equatorial plane $\theta = \pi/2$ of the rotating black hole, and its size depends on the scale of quantum gravity.

> - 对于旋转时空，会出现特征性的“凹痕”[85,89-95]。[5] 该凹痕在旋转黑洞的赤道面 $\theta = \pi/2$ 上更明显，其大小依赖于量子引力能标。

If quantum gravity sets in at Planckian scales, as is expected on dimensional grounds, these deviations will be practically unobservable (see also [10,96]). Importantly, under certain assumptions [89], these effects seem to be a general feature of black holes beyond general relativity [85,89-95].

> 如果如量纲分析预期的那样，量子引力在普朗克尺度生效，这些偏差实际上无法观测 (另见文献 [10,96]。重要的是，在特定假设下 [89]，这些效应似乎是广义相对论之外黑洞的普遍特征 [85,89-95]。

Gravitational Collapse and Improved Buchdahl Limit

引力坍缩与改进的布赫达尔极限

So far we have focused on static black hole configurations and their evaporation. A key ingredient discriminating between physical and unphysical configurations is their dynamics and specifically their formation via a physical process. Among the possible formation processes, gravitational collapse is of utmost importance in the context of astrophysical black holes. Yet, there is no consensus on how to precisely model the collapse - not even at a classical level - and several models have been developed. In the following, we shall summarize some of them, together with their quantum-corrected versions and the corresponding physical implications [97-105]. In particular, the focus of the discussions will be (i) the potential avoidance of singularities when dynamics is accounted for, (ii) the impact of quantum corrections on Penrose’s cosmic censorship conjecture, and (iii) how the Buchdahl’s limit is modified by gravitational antiscreening. As for point (i) , two classes of RG-improved models have been considered in the literature:

> 至此我们一直关注静态黑洞构型及其蒸发。区分物理与非物理构型的关键要素是其动力学，尤其是通过物理过程形成的过程。在所有可能的形成过程中，引力坍缩在天体物理黑洞研究中至关重要。但目前对如何精确建模坍缩——即使是经典层面——仍未达成共识，学界已经发展出多种模型。下文我们将总结其中部分模型，同时介绍它们的量子修正版本和对应的物理推论 [97-105]。具体而言，讨论的重点将是:(i) 考虑动力学后奇点是否有可能避免，(ii) 量子修正对彭罗斯宇宙监督猜想的影响，(iii) 引力反屏蔽如何修改布赫达尔极限。针对第 (i) 点，文献中已经研究了两类 RG 改进模型:

- Those where one fixes the radial dependence of the Newton coupling $G(r)$ from the RG improvement of the classical static system (e.g., to be of the Bonanno-Reuter type) and subsequently studies the dynamics of the mass $m(v)$ induced by the different radial dependence. We shall refer to these models as partially dynamical RG improvements. Examples are those in [98-101] and we will discuss them first.

> - 这类模型先通过经典静态系统的 RG 改进确定牛顿耦合 $G(r)$ 的径向依赖关系 (例如采用博南诺-罗伊特型形式)，随后研究不同径向依赖诱导的质量 $m(v)$ 动力学。我们将这类模型称为部分动力学 RG 改进。[98-101] 中的例子属于这类，我们会先讨论它们。

- Those where the dynamics is encoded in an effective Newton coupling $G(r, v)$ that depends on both the radial and the time coordinates and whose specific analytical form is derived from the *RG* improvement of the classical dynamical system. We will call models belonging to this class fully dynamical RG improvements. Examples are those in [97, 102-105] and we will discuss them next.

> - 这类模型将动力学编码在有效牛顿耦合 $G(r, v)$ 中，该耦合同时依赖径向坐标与时间坐标，其具体解析形式由经典动力学系统的 *RG* 改进推导得到。我们将这类模型称为完全动力学 RG 改进。[97, 102-105] 中的例子属于这类，我们随后会讨论它们。

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[5] The dent introduced in [91-95] and the one discussed in [85,89,90] are however structurally different, since in the case of [85,89,90] the scale identification introduces an angular dependence in all metric components that breaks a mathematical property known as ”circularity.” The non-circularity yields a dent whose boundary has a concave piece, as opposed to the one in [91-95] which is fully convex.

> [5] [91-95] 中提出的凹坑与 [85,89,90] 中讨论的凹坑在结构上不同，因为在 [85,89,90] 的情况中，标度识别给所有度规分量引入了角度依赖，破坏了名为“圆性”的数学性质。非圆性会产生边界带一段凹弧的凹坑，而 [91-95] 中的凹坑边界是完全凸的。

The discussions (ii) and (iii) will only involve the second class of models [102-105].

> 对 (ii) 和 (iii) 的讨论仅涉及第二类模型 [102-105]。

Singularity Resolution and Partially Corrected Collapse Dynamics [98-101]

奇点解决与部分修正坍缩动力学 [98-101]

The models in [98-101], within different classical collapse models and approximations, describe the collapsing spacetime with a homogeneous interior surrounded by a Bonanno-Reuter exterior, which is inserted ad hoc.

> 文献 [98-101] 中的模型在不同经典坍缩模型与近似下，描述了内部均匀、外部为临时插入的邦南诺-罗伊特 (Bonanno-Reuter) 度规的坍缩时空。

In the literature, models of radiating collapsing stars haven been considered, starting from the pioneering work of Oppenheimer and Snyder [106] to the more realistic Vaidya models [107], accounting for the outgoing incoherent radiation and eventually radiating away all the star mass [108]. These models were soon abandoned due to the observation that the radiation emitted undergoes large backreaction effects due to spacetime curvature, and it is infinitely blueshifted in the limit where the mass gets small [109]. In this regime the Vaidya solution is no longer a good approximation, as the large amount of backscattered ingoing radiation ought to be accounted for, and is instead neglected by the outgoing Vaidya model. One may however ask the question of whether a modified exterior model reducing the growth of the curvature as $r \to 0$ can avoid this conclusion.

文献中早已对辐射坍缩恒星模型展开研究，从奥本海默与斯奈德的开创性工作 [106]，到更符合实际的瓦伊达 (Vaidya) 模型 [107]，这类模型考虑了外向非相干辐射，最终会将恒星全部质量辐射出去 [108]。但这些模型很快被弃用，因为研究发现: 时空曲率会对出射辐射产生强烈的反作用，且当质量趋近于零时，辐射会发生无限蓝移 [109]。在该区域瓦伊达解不再是良好近似——大量后向散射的入射辐射本应纳入考虑，却被外向瓦伊达模型忽略。不过我们仍可以追问: 修改后的外部模型若能让曲率增长如 $r \to 0$ ，是否能避开这一结论?

Inspired by the work of Bonanno and Reuter that we described in section ”The Spherically Symmetric, Asymptotically Flat Case,” [98] considered an improved outgoing Vaidya solution

受邦南诺和罗伊特工作的启发，我们在章节“球对称渐近平直情形”中介绍过，文献 [98] 考虑了一种改进的外向瓦伊达解

$$\mathrm{d}s^2 = -\left(1 - \frac{2m(u)\,G(r)}{r}\right)\mathrm{d}u^2 - 2\,\mathrm{d}u\,\mathrm{d}r + r^2\,\mathrm{d}\Omega^2, \tag{44}$$

with $G(r)$ being the Bonanno-Reuter effective Newton coupling (14), to describe the exterior region of a collapsing radiating star. Thanks to the regularity of the Bonanno-Reuter lapse function and of its derivatives, the curvature does not grow to infinite as $r \to 0$, and the problem of the unbound backscattered radiation may be avoided. To show this, it suffices to integrate the radial null geodesics of a backscattered test field:

其中 $G(r)$ 为邦南诺-罗伊特有效牛顿耦合 (14)，用于描述辐射坍缩恒星的外部区域。由于邦南诺-罗伊特时移函数及其导数具有正则性，曲率不会如 $r \to 0$ 增长至无穷，无界后向散射辐射的问题或可避免。要证明这一点，只需对后向散射测试场的径向类光测地线积分:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\lambda^2} + \frac{rmG' - mG}{r^2}\left(\frac{\mathrm{d}u}{\mathrm{d}\lambda}\right)^2 = 0, \quad \frac{\mathrm{d}^2r}{\mathrm{d}\lambda^2} + \frac{m_{,u}G}{r}\left(\frac{\mathrm{d}u}{\mathrm{d}\lambda}\right)^2 = 0. \tag{45}$$

Classically, the problem of unboundness of backscattered radiation comes from the behavior of $\partial_\lambda u$, which diverges in the limit $r \to \infty$. A first integration of Eq. (45) yields the general expression

经典情况下，后向散射辐射无界的问题源于 $\partial_\lambda u$ 的行为——它在 $r \to \infty$ 极限下发散。对式 (45) 完成第一次积分后可得通式

$$\frac{\mathrm{d}u}{\mathrm{d}\lambda} = A\exp\left(\int \frac{Gm - rmG'}{r^2}\,\mathrm{d}u\right), \tag{46}$$

with A being an integration constant. Inserting the specific expression of the Bonanno-Reuter effective Newton coupling, one can easily see that $\lim_{r\to 0}\partial_\lambda u = \mathrm{const}$ [98].

其中 A 是积分常数。代入邦南诺-罗伊特有效牛顿耦合的具体表达式，很容易看出 $\lim_{r\to 0}\partial_\lambda u =$ 为常数 [98]。

Within these partially dynamical RG-improved models, the absence of shell-focusing singularities is inherited by the Bonanno-Reuter scaling of $G(r)$. This has been checked both within the collapse model we just discussed, first analyzed in [98], and via the more known Lemaitre-Tolman-Bondi (LTB) model [99]. The latter is a class of spherically symmetric solutions consisting of non-interacting particles (named ”dust”), where the gravitational collapse and the formation of a singularity is classically unavoidable. To discuss singularity resolution in this case, [99] modeled the star as an homogeneous interior - a spherically symmetric collapsing objects, parameterized as a sphere of non-interacting dust particles - surrounded by an RG-improved exterior of the Bonanno-Reuter type. Due to the matching conditions at the star surface, key properties of the Bonanno-Reuter scaling function are inherited by the interior metric. Specifically, the class of improved dust interiors in geodesic coordinates is

在这些部分动力学重整化群改进模型中，壳聚焦奇点的消除源自 $G(r)$ 的邦南诺-罗伊特标度行为。这一点已在我们刚才讨论的、最早由文献 [98] 分析的坍缩模型中，以及更知名的勒梅特-托尔曼-邦迪 (Lemaitre-Tolman-Bondi, LTB) 模型 [99] 中得到验证。经典 LTB 模型是一类由无相互作用粒子 (称为“尘埃”) 构成的球对称解，经典情况下引力坍缩必然发生，奇点必然形成。为讨论这类情况中的奇点解决，文献 [99] 将恒星建模为内部均匀 (即由无相互作用尘埃粒子构成的球对称坍缩天体)、外部为邦南诺-罗伊特型重整化群改进度规的结构。根据恒星表面的匹配条件，内部度规会继承邦南诺-罗伊特标度函数的核心性质。具体来说，测地坐标系下改进型尘埃内部的类别为

$$ds^2_{\text{int}} = -d\tau^2 + \frac{R'(\tau,r)^2}{1+2E(r)}dr^2 + R(\tau,r)^2 d\Omega^2, \tag{47}$$

where $E(r)$ is an arbitrary function stemming from a partial integration of one of the matching conditions. It represents the total energy per unit mass of the dust particles within a shell of radius r . Moreover, the same matching condition defines the function R as the solution

其中 $E(r)$ 是对其中一个匹配条件做部分积分后得到的任意函数，代表半径为 r 的壳层内尘埃粒子单位质量的总能量。此外，同一个匹配条件将函数 R 定义为下述方程的解

$$\frac{\dot{R}^2}{2} = \frac{G(r)M(r)}{R} + E(r), \tag{48}$$

where $G(r)$ is the Bonanno-Reuter effective Newton coupling and $M(r)$ is the total mass within a sphere of radius r . One can at this point demonstrate that if $R' \neq 0$, the energy-momentum tensor of the improved solutions is bounded. As a consequence, the geodesics of dust particles undergo a bounce in the interior region. In addition, the scalar invariants are finite because of the Bonanno-Reuter scaling. The typical shell-focusing singularities of classical LTB models are thus not formed [99].

其中 $G(r)$ 是邦南诺-罗伊特有效牛顿耦合，$M(r)$ 是半径为 r 的球体内的总质量。此时可以证明: 若 $R' \neq 0$ ，则改进解的能量动量张量有界。因此，尘埃粒子的测地线会在内部区域发生反弹。此外，由于邦南诺-罗伊特标度行为，标量不变量都是有限的。因此经典 LTB 模型典型的壳聚焦奇点不会形成 [99]。

The result is stable under the inclusion of backreaction effects due to Hawking radiation. This has been checked both using an ingoing Eddington-Finkelstein model which parametrizes the ingoing negative energy flux of Hawking radiation [100] and via a standard LTB model [101]. In the latter case though, in order to avoid the shell focusing due to the propagation of ingoing shells of collapsing matter and bouncing shells of backscattered matter, it is key that the collapse occurs fast enough [101]. This allows the final object to be void of singularities.

该结果在纳入霍金辐射引发的反作用效应后仍是稳定的。无论是借助对霍金辐射入射负能通量参数化的入射爱丁顿-芬克斯坦模型 [100]，还是通过标准 LTB 模型 [101]，都验证了这一点。不过在后一种情形中，为了避免坍缩物质入射壳层与背散射物质反弹壳层传播导致的壳层聚焦，坍缩发生得足够快是关键 [101]。这能保证最终形成的天体不存在奇点。

Conditions on Singularity Avoidance from the Collapse of Dust Shells [97]

尘壳坍缩带来的奇点规避条件 [97]

While singularities appear to be avoided when improving classical static solutions, it is not obvious that dynamical black holes formed from a gravitational collapse will be singularity-free. To determine the impact of the collapse dynamics on singularity resolution, [97] studied quantum corrections to dynamical black hole solutions within one of the simplest collapse models - the Tolman-Lemaitre-Oppenheimer-Snyder model [106] - via a fully dynamical RG improvement. The latter aims at describing the collapse of a thin shell of dust with areal radius $R(r,\tau)$, with τ being the proper time of comoving observers along the paths of constant r . The exterior and interior geometries are described by the line elements

尽管在改进经典静态解时奇点似乎可以被规避，但由引力坍缩形成的动力学黑洞是否一定无奇异性尚不明确。为了确定坍缩动力学对奇点解析处理的影响，文献 [97] 通过全动力学重整化群改进，在最简单的坍缩模型之一——托尔曼-勒梅特-奥本海默-斯奈德模型 [106]——中研究了动力学黑洞解的量子修正。该模型旨在描述面积半径为 $R(r,\tau)$ 的薄尘埃壳的坍缩，对共动观测者而言，τ 是其沿恒定 r 路径运动的固有时。外部与内部几何由如下线元描述

$$\mathrm{d}s^2_{\text{ext}} = -\left(1-\frac{2mG_0}{R}\right)\mathrm{d}t^2 + \left(1-\frac{2mG_0}{R}\right)^{-1}\mathrm{d}R^2 + R^2\,\mathrm{d}\Omega^2, \tag{49a}$$

$$\mathrm{d}s^2_{\text{int}} = -\mathrm{d}\tau^2 + \left(R'\right)^2\mathrm{d}r^2 + R^2\,\mathrm{d}\Omega^2. \tag{49b}$$

In particular, the dynamics of R is governed by the classical field equations

特别地，R 的动力学由经典场方程支配

$$\frac{\left(\dot{R}^2 R\right)'}{R^2 R'} = 8\pi G_0 \rho, \tag{50}$$

where $\rho(r,\tau)$ is the dust energy density, which evolves according to the Bianchi identities

其中 $\rho(r,\tau)$ 是尘埃能量密度，它依据比安基恒宣演化

$$\dot{\rho} + \frac{\partial_\tau\left(R^3\right)'}{\left(R^3\right)'}\rho = 0. \tag{51}$$

In these expressions and in the following, a prime denotes derivation with respect to the radial coordinate r , whereas a dot stands for the proper time derivative. Imposing the junction conditions at the boundary of the dust ball, at $r = r_s$, yields the constraint $R(\tau) = a(\tau) r_s$. Correspondingly, the matter energy density becomes $\rho = 3/(8\pi G_0)(\dot{a}/a)^2$, and the areal radius satisfies the equation

在这些表达式以及下文的推导中, 撇号表示对径向坐标 r 求导, 点号表示对固有时求导。在尘埃球边界 $r = r_s$ 处施加连接条件,可得到约束 $R(\tau) = a(\tau) r_s$。相应地,物质能量密度变为 $\rho = 3/(8\pi G_0)(\dot{a}/a)^2$ ，面积半径满足方程

$$\dot{R}^2 = \frac{2}{R}\frac{4\pi G_0}{3}\int_0^r \rho(\tau, x)\left[R^3(\tau, x)\right]' dx. \tag{52}$$

If one turns quantum gravity effects on, the effective dynamics might be described by equations that are structurally similar, but with G_0 replaced by a coordinate-dependent function. In particular, the dominant energy scale of the system in this case is the matter energy density ρ , and thus the classical field equation is replaced by an effective one:

如果引入量子引力效应，有效动力学可以用结构相似的方程描述，只需要将 G_0 替换为依赖坐标的函数。具体而言，此时系统的主导能量标度是物质能量密度 ρ ，因此经典场方程被替换为有效场方程:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi\rho}{3} G(\rho). \tag{53}$$

The classical Bianchi identities remain instead unmodified. Before proceeding, we note that the procedure exploited above and drawn from [97] is an RG improvement at the level of the field equation. As it will be clarified later on, in section "Improving the RG Improvement: Methods and Physical Results," this is less fundamental than an RG improvement at the level of the action, as it disregard additional terms stemming from higher-derivative quantum corrections in the effective action; nonetheless, a scale identification involving the matter energy density ρ is appropriate and consistent with Bianchi identities.

而经典比安基恒等式保持不变。在继续推导之前，我们需要指出，上述来自文献 [97] 的方案是场方程层面的重整化群改进。正如我们将在后文“改进重整化群改进: 方法与物理结果”一节中阐明的，这种方法的基础性弱于作用量层面的重整化群改进，因为它忽略了有效作用量中来自高阶导数量子修正的额外项；尽管如此，涉及物质能量密度 ρ 的标度设定是合理的，且与比安基恒等式相容。

Combining the field equations for a and ρ yields the integral

结合 a 和 ρ 的场方程可得积分

$$\int_{\rho(\tau_0=0)}^{\rho(\tau)} \frac{\mathrm{d}\rho}{\sqrt{24\pi\rho^3 G(\rho)}} = \tau. \tag{54}$$

If the integral is convergent as $\rho \to \infty$, then there exist geodesics attaining the singularity at a finite proper time τ_* . Singularity avoidance thus requires the divergence of the integral, and this can only occur if the fall off of $G(\rho)$ is sufficiently rapid. Concretely, close to the classical singularity $G_k \sim k^{-\alpha}$, and one can assume $k^2 \sim G_k \rho$ [97, 110] , implying that

如果当 $\rho \to \infty$ 时该积分收敛，那么就存在类测地线在有限固有时 τ_* 抵达奇点。因此规避奇点要求积分发散，而这只有当 $G(\rho)$ 的衰减足够快时才能实现。具体而言，在靠近经典奇点 $G_k \sim k^{-\alpha}$ 处，可以假设 $k^2 \sim G_k\rho$ [97, 110]，由此可得

$$\rho(\tau \to \tau_*) \sim (\tau_* - \tau)^{-(2+\alpha)}, \tag{55a}$$

$$a(\tau \to \tau_*) \sim (\tau_* - \tau)^{(2+\alpha)/3}, \tag{55b}$$

$$k(\tau \to \tau_*) \sim (\tau_* - \tau)^{-1}, \tag{55c}$$

$$G(\tau \to \tau_*) \sim (\tau_* - \tau)^{\alpha}. \tag{55d}$$

If quantum gravity remains perturbative across all scales, i.e., $k^2 G_k \ll 1$, then $\alpha \geq 2$ and the classical singularity is not resolved by the effective modifications above. Therefore, as a general result, the collapse dynamics makes singularity resolution less straightforward than in the classical, static case and requires stronger deviations from the classical dynamics, which go beyond the regime where perturbation theory holds. This general expectation is also met by the concrete model devised in [102- 104] that we review in the following in relation to the cosmic censorship conjecture.

如果量子引力在所有标度下都保持微扰，即 $k^2 G_k \ll 1$，那么有 $\alpha \geq 2$，经典奇点无法通过上述有效修改解决。因此，一个普遍结论是，坍缩动力学让奇点解析处理比经典静态情形更困难，要求对经典动力学有更大偏离，这种偏离超出了微扰论成立的范围。我们接下来会结合宇宙监督假设回顾文献 [102-104] 构建的具体模型，该模型也符合这一普遍预期。

Gravitational Collapse and Cosmic Censorship in Asymptotic Safety [102-104] After the formulation of the singularity theorems [111], Penrose conjectured that curvature singularities ought to always be hidden behind an event horizon [112] - a ”cosmic censor” preventing far-away physicists from seeing their theories breaking down and physics losing its predictive power.

渐近安全中的引力坍缩与宇宙监督假设 [102-104] 在奇点定理 [111] 提出后，彭罗斯猜想曲率奇点应当永远被事件视界隐藏 [112]——存在一位“宇宙监督”，防止远处的物理学家观测到理论失效、物理丧失预测能力的现象。

In order to test Penrose’s conjecture, [102-104] studied the dynamical process of black hole formation and discussed the singularity structure as well as the dynamical evolution of the event horizon within a quantum gravity-corrected Vaidya-Kuroda-Papapetrou (VKP) model [113, 114].

为了检验彭罗斯猜想，文献 [102-104] 研究了黑洞形成的动力学过程，并在量子引力修正的 Vaidya-Kuroda-Papapetrou(VKP) 模型 [113, 114] 框架下讨论了奇点结构以及事件视界的动力学演化。

The classical VKP model describes the gravitational collapse through an ingoing Vaidya metric

经典 VKP 模型通过入射 Vaidya 度规描述引力坍缩

$$ds^2 = -f(r,v)\,dv^2 + 2dvdr + r^2 d\Omega^2, \tag{56}$$

with a lapse function that depends on the advanced time v via the dynamical mass $m(v)$

其延迟函数通过动力学质量 $m(v)$ 依赖于超前时间 v

$$f_{cl}(r,v) = 1 - \frac{2m(v)\,G_0}{r}. \tag{57}$$

In this model the spacetime is initially (for $v \le 0$) a flat Minkowski background; subsequently, radiation from a nearby massive star is focused toward $r = 0$. This is modeled by a set of ingoing radial null geodesics that are focused to $r = 0$ and raise the black hole mass from zero at $v = 0$ to $m(v)$. Such a shell-focusing classically cause the formation of a curvature singularity at $r = 0$. The collapse ends when all radiation from the star is radiated away, and $m(v)$ reaches a final constant value, $m(v) = m$. The resulting spacetime is thus a Schwarzschild black hole with mass m . Within the VKP model, the growth of the mass function is fixed to be linear, $m(v) = \lambda v$, for $v \in [0, \bar{v}]$, with $\bar{v}$ denoting the advance time at the end of the collapse.

在该模型中，时空初始时 (对于 $v \le 0$) 是平直闵氏背景；随后，邻近大质量恒星的辐射向 $r = 0$ 汇聚。这一过程由一组入射径向类光测地线建模，这些测地线汇聚到 $r = 0$ ，使黑洞质量从 $v = 0$ 处的零增长到 $m(v)$ 。经典层面上这类壳汇聚会在 $r = 0$ 处形成曲率奇点。当恒星的所有辐射都辐射完毕后，坍缩结束，$m(v)$ 达到最终恒定值 $m(v) = m$ 。最终形成的时空就是质量为 m 的史瓦西黑洞。在 VKP 模型中，质量函数被固定为线性增长，即 $m(v) = \lambda v$ ，适用于 $v \in [0, \bar{v}]$ ，其中 $\bar{v}$ 表示坍缩结束时的超前时间。

The procedure of RG improvement yields in this case a generalized Vaidya spacetime [115]

在此情形下，重整化群改进方法得到了广义 Vaidya 时空 [115]

$$f(r,v) = 1 - \frac{2M(r,v)}{r}, \tag{58}$$

with generalized mass function $M(r,v) = m(v)\,G(r,v)$. Generalized Vaidya spacetimes [115] are a class of solutions to Einstein-like field equation with an effective energy-momentum tensor having both a null and a non-null component. It reads

其广义质量函数为 $M(r,v) = m(v)\,G(r,v)$ 。广义 Vaidya 时空 [115] 是类爱因斯坦场方程的一类解，它具有同时包含类零分量和非零分量的有效能量动量张量，形式如下

$$T_{\mu v} = \mu l_\mu l_v + (\rho + p)\left(l_\mu n_v + l_v n_\mu\right) + p g_{\mu v}, \tag{59}$$

where l_μ and n_μ are null vectors satisfying the condition $l_\mu n^\mu = -1, \mu$ is the radiation energy density provoking the mass variation of the black hole,

其中 l_μ 和 n_μ 是满足条件 $l_\mu n^\mu = -1, \mu$ 的零矢量，$l_\mu n^\mu = -1, \mu$ 是引发黑洞质量变化的辐射能量密度，

$$\mu(r,v) = \frac{1}{4\pi G_0 r^2} \frac{\partial M(r,v)}{\partial v}, \tag{60}$$

whereas

而

$$\rho(r,v) = \frac{1}{4\pi G_0 r^2} \frac{\partial M(r,v)}{\partial r}, \; p(r,v) = -\frac{1}{8\pi G_0 r} \frac{\partial^2 M(r,v)}{\partial r^2}, \tag{61}$$

are the energy density and pressure associated with the non-null fluid sourcing the effective modification to Einstein equation due to the replacement $G_0 \to G(r,v)$.

是与非零流体相关的能量密度和压强，该非零流体是替换 $G_0 \to G(r,v)$ 对爱因斯坦方程产生有效修正的源。

In an RG-improved model $G(r,v) \equiv G(k[r,v])$, with the functional form of G given by Eq. (9). The cutoff identification considered in [102-104] is $k^4 \sim \rho$, ρ being the energy density of the ingoing radiation. Indeed, since the black hole interior is homogeneous and formed by radiation, one can exploit the relation $\mu_{cl} \sim T^4 \sim k^4$ [15], where μ_{cl} is the energy density in the classical ($G(r,v) = G_0$) Vaidya model:

在重整化群改进模型 $G(r,v) \equiv G(k[r,v])$ 中，G 的函数形式由式 (9) 给出。文献 [102-104] 中考虑的截断标识为 $k^4 \sim \rho$ ，ρ 是入射辐射的能量密度。实际上，由于黑洞内部均匀且由辐射构成，可以利用关系 $\mu_{cl} \sim T^4 \sim k^4$ [15]，其中 μ_{cl} 是经典 ($G(r,v) = G_0$) Vaidya 模型中的能量密度:

$$\mu_{cl}(r) = \frac{\dot{m}(v)}{4\pi r^2} = \frac{\lambda}{4\pi r^2}. \tag{62}$$

The resulting RG-improved lapse function reads [102]

最终得到的重整化群改进延迟函数为 [102]

$$f_{qu}(r,v) = 1 - \frac{2m(v)\,G_0}{r + \alpha\sqrt{\lambda}}, \tag{63}$$

with $\alpha \equiv g_*^{-1} G_0/\sqrt{4\pi}$. To leading order, the Ricci and Kretschmann scalars scale as $\sim 1/r^2$ and $\sim 1/r^4$, respectively, indicating a weakening of the classical singularity. Specifically, the singularity at $r = 0$ turns out to be gravitationally weak (or integrable, according to Tipler classification [116]): curvature invariants diverge as $r \to 0$, but the geodesic equation can be integrated, so that the spacetime is geodesically complete. The case of black holes with integrable singularities, as the one derived in [102] and further explored in [103,104], is particularly important, as it could provide a consistent alternative to regular black holes whose inner horizon is unstable [64-71].

其中 $\alpha \equiv g_*^{-1} G_0/\sqrt{4\pi}$ 。到领头阶，里奇标量和克雷奇曼标量分别按 $\sim 1/r^2$ 和 $\sim 1/r^4$ 标度，这表明经典奇点被弱化。具体而言，$r = 0$ 处的奇点被证明是引力弱奇点 (根据蒂普勒分类 [116] 也称为可积奇点): 曲率不变量按 $r \to 0$ 发散，但测地线方程可以积分，因此时空是测地完备的。如文献 [102] 推导、并经 [103,104] 进一步研究得到的带可积奇点的黑洞这类情形格外重要，因为它可以为内视界不稳定的正则黑洞 [64-71] 提供自洽替代方案。

Due to the additional term in the denominator of the lapse function, the apparent horizon (AH) gets shifted with respect to the classical case:

由于延迟函数分母中存在额外项，表观视界 (AH) 相对经典情形发生偏移:

$$r_{AH} = 2m(v)\,G_0 - \alpha\sqrt{\lambda}. \tag{64}$$

Although in the case of dynamical spacetimes the AH does not coincide with the event horizon (EH), the above expression indicates that in the quantum-corrected spacetime, horizons tend to form later than in the classical version. Specifically, the minimum irradiation period required to form a horizon is $v_{\min} = \left(2g_*\sqrt{4\pi\lambda}\right)^{-1}$.

尽管在动态时空的情况下表观视界 AH 与事件视界 EH 并不重合，但上述表达式表明，在量子修正时空中，视界的形成时间晚于经典情形。具体而言，形成一个视界所需的最小照射周期为 $v_{\min} = \left(2g_*\sqrt{4\pi\lambda}\right)^{-1}$。

Similarly to the classical collapse [113, 114], there exists a critical irradiation rate λ_c below which the singularity is formed before the EH. In this case the singularity is globally naked, and the first null ray departing from the singularity and reaching future null infinity coincides with a Cauchy horizon. The overall causal structure of the RG-improved spacetime for $\lambda \leq \lambda_c$ is drawn in Fig. 7.

与经典坍缩 [113, 114] 类似，此处存在一个临界照射率 λ_c，低于该值时，奇点会在事件视界 EH 形成之前产生。这种情况下奇点是全局裸奇点，从奇点出发、到达未来类光无穷远的第一束类光射线与柯西视界重合。参数为 $\lambda \leq \lambda_c$ 时，重整化群改进时空的整体因果结构如图 7 所示。

A striking feature of the RG-improved model is that the critical infusion rate λ_c is higher than in the classical case. Therefore, the formation of naked singularities and the corresponding violation of Penrose's cosmic censorship conjecture seem to be favored by gravitational antiscreening. This can be understood intuitively, insofar as a weakening of gravity toward high energies can delay the formation of horizons. At the same time, the replacement of the classical singularity with an integrable one might render the presence of cosmic censor unnecessary.

重整化群改进模型一个显著的特点是，临界注入率 λ_c 高于经典情形下的对应值。因此，引力反屏蔽似乎更易导致裸奇点形成，同时相应地违反彭罗斯宇宙监督猜想。这一点可以直观理解: 引力在高能区会减弱，因此能够推迟视界的形成。与此同时，用可积奇点替换经典奇点或许意味着宇宙监督假设并非必要。

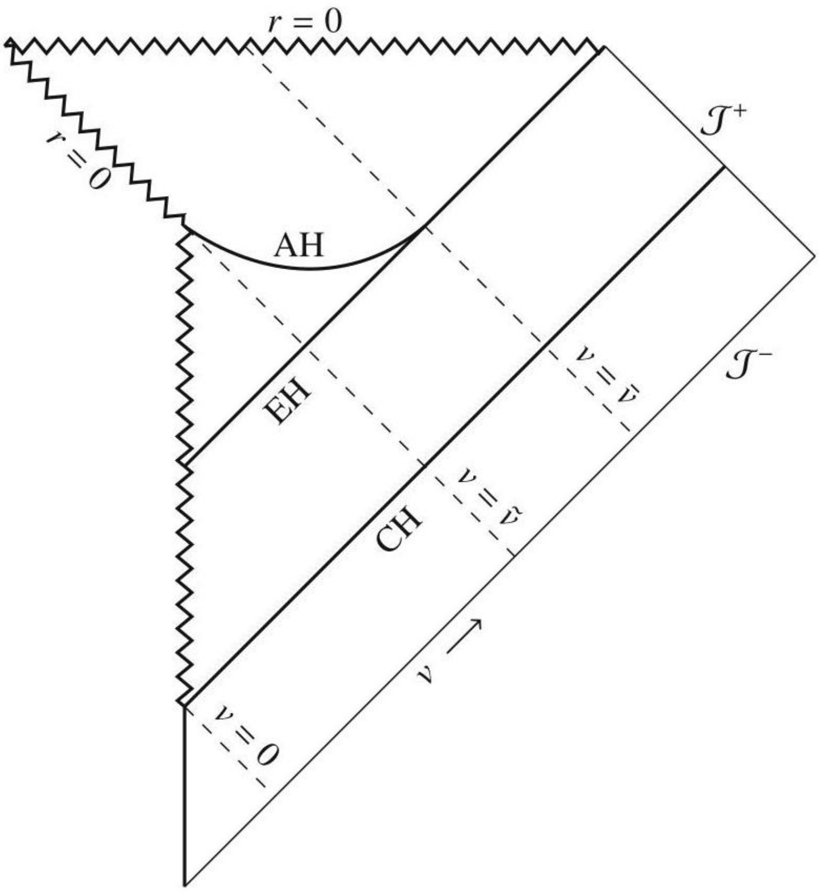


Fig. 7 Penrose diagram of the dynamical spacetime with improved lapse function (63) for $\lambda \leq \lambda_c$

图 7 参数为 $\lambda \leq \lambda_c$ 时，带改进 lapse 函数 (63) 的动态时空的彭罗斯图

Stellar Mass-Radius Relation and Improved Buchdahl Limit [105]

恒星质量-半径关系与改进的布赫达赫极限 [105]

A broader, if perhaps less detailed way to investigate possible endpoints of the gravitational collapse of massive stars is to analyze the equilibrium configuration of massive astrophysical objects. This is classically determined by the so-called Tolman-Oppenheimer-Volkoff (TOV) stellar equilibrium equation [117, 118]. The derivation of the latter goes as follow. One models a star as a self-gravitating perfect fluid with proper energy density $\varepsilon(r)$ and pressure determined by an equation of state $p = p(\varepsilon)$. The star energy-momentum thus reads

研究大质量恒星引力坍缩可能终态的一种更宽泛 (或许不够精细) 的方法是分析大质量天体的平衡构型。经典情况下这由著名的托尔曼-奥本海默-沃尔科夫 (TOV) 恒星平衡方程确定 [117, 118]。该方程推导过程如下: 将恒星建模为自引力理想流体, 其固有能量密度为 $\varepsilon(r)$, 压强由物态方程 $p = p(\varepsilon)$ 确定。因此恒星的能量动量张量可写为

$$T_{\mu\upsilon} = (\varepsilon + p(\varepsilon))\, c^{-2} u_{\mu} u_{\upsilon} - p(\varepsilon)\, g_{\mu\upsilon}, \tag{65}$$

and enters the right-hand side of the Einstein equations. Inserting in the resulting modified equations, the expression for the most general static and spherically symmetric metric and combining it with the Bianchi identities gives the TOV equation [117, 118]

并代入爱因斯坦方程的右侧。将最一般的静态球对称度规表达式代入得到的修正方程，再结合比安基恒等式，即可推导出 TOV 方程 [117, 118]

$$p'(r) = -(p(r) + \varepsilon(r)) \frac{G_0}{c^2 r^2} \left(M(r) + \frac{4\pi r^3}{c^2} p(r) \right) \left(1 - \frac{2G_0 M(r)}{c^2 r} \right)^{-1}, \tag{66}$$

where $M(r)$ denotes the mass contained within a sphere of radius r

其中 $M(r)$ 表示半径为 r 的球内包含的质量

$$M(r) = \frac{4\pi}{c^2} \int_0^r \varepsilon(x)\, x^2 \,\mathrm{d}x, \tag{67}$$

so that for a star of radius R_* the total mass is $M_* = M(R_*)$. One can see that in the non-relativistic limit, one recovers the Newtonian equation of hydrostatic equilibrium.

因此对于半径为 R_* 的恒星，总质量为 $M_* = M(R_*)$ 。可以看到，在非相对论极限下，我们可以重新得到牛顿流体静力学平衡方程。

The importance of the TOV equation in the context of black hole physics lies in the possibility to extract a crucial theoretical bound - known as the Buchdahl limit [119] - that regulates the maximal sustainable density of massive stars. Beyond the critical limit, the stellar equilibrium is broken, and the star fluid starts collapsing under its own pressure. To see how this limit emerges classically, let us consider an incompressible fluid with constant proper energy density $\varepsilon_0 = \rho_0 c^2$. In this case $M(r) = M_* r^3/R_*^3$, and the TOV equation can be integrated together with the boundary condition $p(R_*) = 0$, resulting in an expression for the central pressure:

TOV 方程在黑洞物理研究中的重要性在于，我们可以从中得到一个关键理论边界——即布赫达赫极限 [119]，它约束了大质量恒星能维持的最大密度。超过该临界极限后，恒星平衡被打破，恒星流体会在自身压强作用下开始坍缩。下面我们来看经典情况下该极限是如何得到的: 考虑一个固有能量密度恒定的不可压缩流体 $\varepsilon_0 = \rho_0 c^2$ 。此时 $M(r) = M_* r^3/R_*^3$ ，结合边界条件 $p(R_*) = 0$ 对 TOV 方程积分，可得到中心压强的表达式:

$$p(0) = -\frac{3c^2 M_* \left(3G_0 M_* + c^2 R_* \left(\sqrt{1 - \frac{2G_0 M_*}{R_* c^2}} - 1 \right) \right)}{4\pi R_*^3 \left(4c^2 R_* - 9G_0 M_*\right)}. \tag{68}$$

The condition that the central pressure $p(0)$ be finite and positive finally yields the classical Buchdahl bound for stellar equilibrium:

中心压强 $p(0)$ 有限且为正的条件，最终给出恒星平衡的经典布赫达赫界:

$$M_* < \frac{4c^2}{9G_0} R_*. \tag{69}$$

If the bound is violated, the configuration is unstable, and the object starts collapsing; if no mechanism exists that halts the collapse, the star surface will eventually cross its Schwarzschild radius, thus resulting in a black hole (cf. left panel of Fig. 8).

如果该边界被打破，构型不稳定，天体开始坍缩；若不存在阻止坍缩的机制，恒星表面最终会穿过史瓦西半径，形成黑洞 (参见图 8 左图)。

The question of whether and how the Buchdahl limit is affected by gravitational antiscreening has been tackled in [105]. The investigation is based on the RG improvement but follows the idea, first proposed by Markov and Mukhanov [120], that the antiscreening character of gravity ought to be included via an energy-dependent Newton coupling that is introduced as an effective multiplicative coupling between matter and geometry. At the level of the action, this boils down to introducing an energy-dependent coupling $\chi(\varepsilon)$, so that

引力反屏蔽是否以及如何影响布赫达赫极限的问题已在文献 [105] 中得到研究。该研究基于重整化群改进，遵循马尔可夫和穆哈诺夫最早提出的观点 [120]: 引力的反屏蔽特性应当通过依赖能量的牛顿耦合引入，即作为物质与几何之间的有效乘法耦合。在作用量层面，这等价于引入依赖能量的耦合 $\chi(\varepsilon)$ ，因此

$$\Gamma_0 = \frac{1}{16\pi G_0} \int \mathrm{d}^4x \sqrt{-g} \left(R + 2\chi(\varepsilon) \mathcal{L}_{\text{matter}}\right). \tag{70}$$

Einstein equations are thus modified by an effective energy-momentum tensor

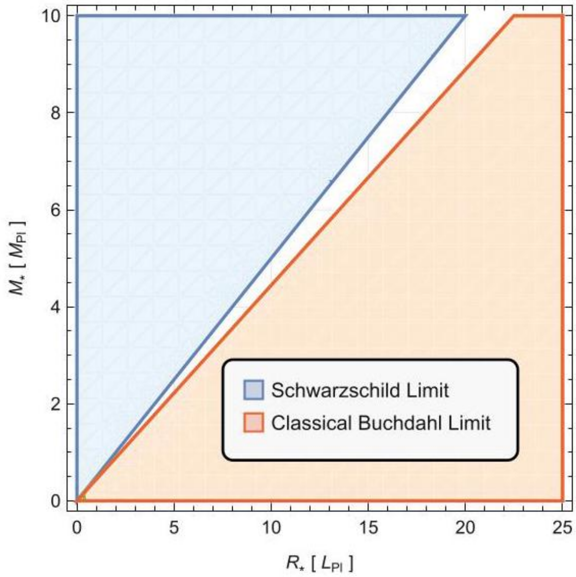
爱因斯坦方程因此被有效能量动量张量修正

$$\Lambda_{\mu\nu} \equiv \left(\varepsilon \frac{\partial \chi}{\partial \varepsilon} + \chi\right) T_{\mu\nu} - \varepsilon^2 \frac{\partial \chi}{\partial \varepsilon} g_{\mu\nu}, \tag{71}$$

from which one can read off an effective, energy-dependent version of the Newton coupling and of the cosmological constant

从中我们可以得到依赖能量的有效形式的牛顿耦合和宇宙学常数

$$G_{\text{eff}}(\varepsilon) \equiv \frac{c^4}{8\pi} \frac{\partial(\varepsilon\chi)}{\partial \varepsilon}, \ \Lambda_{\text{eff}}(\varepsilon) = -\varepsilon^2 \frac{\partial \chi}{\partial \varepsilon}, \tag{72}$$



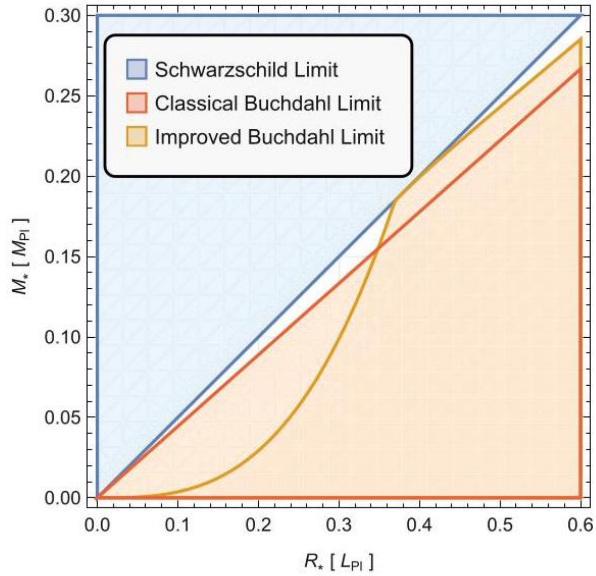


Fig. 8 The figures summarize the findings in [105] and show the Schwarzschild, classical Buchdahl, and improved Buchdahl limits in the (R_*, M_*) plane, using two different zoom levels. On large scales (see left panel), the classical and quantum-improved cases are indistinguishable. Zooming-in on Planckian scales (see right panel), large deviations become apparent: the improved Buchdahl line approaches the Schwarzschild limit, and there exists a critical point at sub-Planckian scales where the two coincides. Below this critical scale, the Buchdahl bound becomes nonlinear, and the formation of black holes from a collapse becomes more likely due to a larger instability region. Notably, this is consistent with a semi-classical treatment of the problem [121]

图 8 本图总结了文献 [105] 的研究结果，在 (R_*, M_*) 平面上通过两种不同缩放比例展示了史瓦西极限、经典布赫达赫极限和改进的布赫达赫极限。在大尺度上 (见左图)，经典情形与量子改进情形无法区分。放大到普朗克尺度 (见右图)，可以看到明显的偏差: 改进后的布赫达赫线趋近于史瓦西极限，在亚普朗克尺度存在一个临界点，两个极限在此处重合。在该临界尺度以下，布赫达赫界变为非线性，且由于不稳定性区域更大，坍缩形成黑洞的可能性更高。值得注意的是，这与该问题的半经典处理结果一致 [121]

akin to the RG-improved ones. Following the same steps as in the classical case and exploiting Eq. (9), one can find the quantum-improved version of the central pressure [105]. It reads

与重整化群改进结果类似。遵循经典情形的相同步骤并利用式 (9)，我们可以得到中心压强的量子改进形式 [105]，其表达式为

$$p(0) = \frac{\varepsilon_{\rm Pl}}{g_*^{-1}} \frac{\mathcal{N}(R_*, M_*)}{\mathcal{D}(R_*, M_*)}, \tag{73}$$

where ε_{Pl} is a Planckian energy density and

其中 ε_{Pl} 是普朗克能量密度，且

$$\mathcal{N} = \left(e^{\frac{c^2}{g_*\varepsilon_{\mathrm{Pl}}}\frac{3M_*}{4\pi R_*^3}}-1\right)\left(g_*^{-1}M_* e^{\frac{c^2}{g_*\varepsilon_{\mathrm{Pl}}}\frac{3M_*}{4\pi R_*^3}} - 2\pi R_*^3 \frac{\varepsilon_{\mathrm{Pl}}}{c^2}\left(e^{\frac{c^2}{g_*\varepsilon_{\mathrm{Pl}}}\frac{3M_*}{4\pi R_*^3}} - 1\right)\right)$$

$$\left(\sqrt{1-\frac{2G_0M_*}{R_*c^2}}-1\right) \tag{74a}$$

$$\mathcal{D} = g_*^{-1}M_* e^{\frac{c^2}{g_*\varepsilon_{\mathrm{Pl}}}\frac{3M_*}{4\pi R_*^3}} + 2\pi R_*^3 \frac{\varepsilon_{\mathrm{Pl}}}{c^2}\left(e^{\frac{c^2}{g_*\varepsilon_{\mathrm{Pl}}}\frac{3M_*}{4\pi R_*^3}} - 1\right)\left(\sqrt{1-\frac{2G_0M_*}{R_*c^2}}-1\right).$$

(74b)

As shown in Fig. 8, the classical and improved Buchdahl bounds coincide at large radii. Nonetheless, substantial deviations become evident in the Planckian region. In there the tilt of the improved Buchdahl limit is smaller than its classical counterpart, so that the quantum Buchdahl line gets arbitrarily close to the Schwarzschild limit. As a consequence, this scenario would predict the existence of ultra-compact horizonless objects of Planckian size, whose compactness is beyond the one established by the classical Buchdahl limit. In particular, there exists a critical point at trans-Planckian scales, $R_{cr} \sim 0.37 g_*^{-1/2} l_{Pl}$, where the Schwarzschild and improved Buchdahl limit coincide. The classical case, where this critical point is absent, is recovered as the fixed-point g_* is pushed to infinity. Beyond the critical point, the Buchdahl inequality undergoes a major change, indicating a potential transition to a quantum gravity-dominated phase. In such a phase, the function $\mathcal{D}$ turns negative, and the Buchdahl limit is determined by the condition $\mathcal{N} < 0$, which yields a cubic relation

如图 8 所示，经典与改进后的布查德界限在大半径处重合。但在普朗克区域会出现明显偏差。在该区域内，改进后布查德界限的斜率小于经典情形，因此量子布查德线会任意接近史瓦西界限。由此，该场景预言存在普朗克尺度的超致密无视界天体，其致密程度超过经典布查德界限给出的范围。特别地，在跨普朗克尺度存在一个临界点 $R_{cr} \sim 0.37 g_*^{-1/2} l_{Pl}$ ，史瓦西界限与改进后的布查德界限在此处重合。当不动点 g_* 趋近于无穷时，就会回到不存在该临界点的经典情形。临界点之外，布查德不等式发生重大改变，表明系统可能跃迁至量子引力主导的相。在该相中，函数 $\mathcal{D}$ 变为负值，布查德界限由条件 $\mathcal{N} < 0$ 确定，由此得到一个三次关系

$$R_* \gtrsim 0.27 g_*^{-1}\left(\frac{M_*}{M_{\mathrm{Pl}}}\right)^{\frac{1}{3}} L_{\mathrm{Pl}}, \tag{75}$$

resembling the scaling relation characterizing ”Planck stars” [122]. Notably, the general picture is compatible with the results obtained within the semi-classical approach in [121].

，其与“普朗克恒星”特征标度关系类似 [122]。值得注意的是，这一整体图像与文献 [121] 中半经典方法得到的结果一致。

Improving the RG Improvement: Methods and Physical Results

改进重整化群改进法: 方法与物理结果

The RG improvement provides a straightforward recipe to determine qualitative features of quantum-corrected black hole (and cosmological) spacetimes - some of which have been summarized in section ”RG-Improved Black Holes.” Yet, as remarked in section ”RG Improvement: Key Idea,” the procedure suffers from a number of problems and ambiguities, making its relation to the asymptotic safety program unsettled. These issues followed several attempts to make the RG improvement procedure more rigorous and to fill the gap between model building and first-principle calculations in quantum gravity. In this section we discuss some of the issues and their proposed partial or complete resolutions.

重整化群改进为确定量子修正黑洞 (及宇宙学) 时空的定性特征提供了一套简便方案——其中部分特征已在“重整化群改进黑洞”一节中总结。但正如“重整化群改进: 核心理念”一节所述，该过程存在诸多问题与歧义，导致它和渐近安全方案的关联仍不明确。针对这些问题，已有多项尝试旨在让重整化群改进过程更严谨，填补量子引力中模型构建与第一性原理计算之间的缺口。本节我们将讨论其中部分问题，以及已提出的部分或完整解决方案。

Constraints on the Cutoff Identification from Bianchi Identities

比安基恒等式对截断识别的约束

One of the main concerns about the application of the RG improvement is the lack of a clear and systematic recipe to determine the map $k \mapsto k(x)$. The authors of [43, 46, 47, 81, 82] have independently shown that some cutoff identifications may be incompatible with diffeomorphism invariance; hence, enforcing the validity of the Bianchi identities could constrain or even completely fix the functional relation $k(x)$. In this subsection we will show how this is implemented through some examples. In doing so, we will follow the review [123] closely.

RG 改进应用的核心问题之一，是缺乏清晰系统的方法来确定映射 $k \mapsto k(x)$ 。[43, 46, 47, 81, 82] 的作者已独立证明，部分截断识别可能与微分同胚不变性不相容；因此，要求比安基恒等式成立可以约束甚至完全确定函数关系 $k(x)$ 。本小节我们将通过实例说明这一约束的实现方式，阐述过程将严格遵循综述文献 [123] 的内容。

To illustrate how the Bianchi identities can constrain the map $k(x)$, let us consider the EAA in the Einstein-Hilbert truncation complemented by a matter Lagrangian $\mathcal{L}_{\text{matter}}$:

为说明比安基恒等式如何约束映射 $k(x)$ ，我们考虑添加了物质拉格朗日量 $\mathcal{L}_{\text{matter}}$ 的爱因斯坦-希尔伯特截断下的有效平均作用量 (EAA):

$$\Gamma_k = \int d^4x \sqrt{-g} \left(\frac{1}{16\pi G_k} (R - 2\Lambda_k) + \mathcal{L}_{\text{matter}} \right). \tag{76}$$

We shall neglect the running of the matter couplings in the following. If $k = k(x)$ (in particular, $k(x) \equiv k_{\text{dec}}(x)$, cf. section ”RG Improvement: Key Idea”), then the classical gravitational field equations will be modified by an effective energy-momentum tensor

下文我们将忽略物质耦合的跑动。若 $k = k(x)$ (具体见“RG 改进: 核心思想”一节的 $k(x) \equiv k_{\text{dec}}(x)$)，经典引力场方程会被有效能动张量修改

$$\Delta t_{\mu\nu} \equiv G_k \left(\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box \right) G_k^{-1}, \tag{77}$$

encoding the vacuum polarization effects of the quantum gravitational field [81]. The resulting field equations thus read

该张量编码了量子引力场的真空极化效应 [81]，因此修改后的场方程形式为

$$G_{\mu\upsilon} = 8\pi G_k T_{\mu\upsilon} - \Lambda_k g_{\mu\upsilon} + \alpha \Delta t_{\mu\upsilon}, \tag{78}$$

where $\alpha = 1$ if the RG improvement is performed at the level of the action, whereas $\alpha = 0$ if it is employed at the level of field equations. Enforcing the validity of the Bianchi identities thus yields the "consistency condition" [43, 46, 47, 81, 82]

其中，如果 RG 改进在作用量层面进行，则 $\alpha = 1$；如果在场方程层面进行，则 $\alpha = 0$。要求比安基恒等式成立可得到“一致性条件”[43, 46, 47, 81, 82]

$$\nabla^{\mu} G_{\mu\upsilon} = \left(8\pi G_k' T_{\mu\upsilon} - \Lambda_k' g_{\mu\upsilon}\right) \nabla^{\mu} k(x) + 8\pi G_k \nabla^{\mu} T_{\mu\upsilon} + \alpha \nabla^{\mu} \Delta t_{\mu\upsilon} = 0,$$

(79)

where a prime denotes differentiation with respect to the RG scale k . This equation can be exploited to constrain or even determine $k(x)$. The practical realization of this constraint strongly depends on whether the RG improvement is performed at the level of the action or elsewhere; we shall discuss these two cases separately.

式中撇号表示对 RG 标度 k 求导。该方程可用于约束甚至确定 $k(x)$。该约束的具体实现强烈依赖于 RG 改进是否在作用量层面进行，我们将分开讨论这两种情况。

RG improving at the level of the action boils down to setting $\alpha = 1$ in Eq. (79). The modified Bianchi identities thus lead to the conditions [47]

在作用量层面进行 RG 改进等价于令 (79) 式中 $\alpha = 1$。修改后的比安基恒等式由此得到条件 [47]

$$\nabla^{\mu} G_{\mu\nu} = +G_k' G_k^{-1} \left\{\left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - 8\pi G_k T_{\mu\upsilon} + \Lambda_k g_{\mu\nu}\right) - R_{\mu\nu}\right\} \nabla^{\mu} k(x)$$

$$+ \left(8\pi G_k' T_{\mu\upsilon} - \Lambda_k' g_{\mu\upsilon}\right) \nabla^{\mu} k(x) + 8\pi G_k \nabla^{\mu} T_{\mu\upsilon} = 0.$$

(80)

This relation further simplifies if the energy flow between the gravitational and matter sector is negligible, as in this case, the matter energy-momentum tensor $T_{\mu\upsilon}$ is separately conserved, $\nabla^{\mu} T_{\mu\nu} \approx 0$. Under this condition, the Bianchi identities yield the constraint [46, 47, 81]

如果引力与物质 sector 之间的能流可忽略，该关系还可进一步简化: 这种情况下物质能动张量 $T_{\mu\upsilon}$ 分别守恒，即 $\nabla^{\mu} T_{\mu\nu} \approx 0$。在此条件下，比安基恒等式给出约束 [46, 47, 81]

$$G_k' R = 2\left(G_k' \Lambda_k - \Lambda_k' G_k\right). \tag{81}$$

At this point the specific form of $k(x)$ depends on the running couplings G_k and Λ_k , which are in turn determined as solutions to the beta functions. As a paradigmatic example, in the fixed-point regime $G_k \sim g_* k^{-2}$ and $\Lambda_k \sim \lambda_* k^2$ so that the above consistency constraint imposes

至此，$k(x)$ 的具体形式依赖于跑动耦合 G_k 和 Λ_k ，而二者本身是贝塔函数的解。作为典型范例，在不动点区域 $G_k \sim g_* k^{-2}$ 和 $\Lambda_k \sim \lambda_* k^2$ ，因此上述一致性约束要求

$$k^2 = \frac{R}{4\lambda_*}. \tag{82}$$

A generalization of Eq. (81) to the case of quadratic gravity was considered in [76], cf. Eq. (31). Notably the result (82) is independent of the specific $f(R)$ truncation, at least when focusing on the fixed-point regime [46, 124]. The relation (82) also has important implications in cosmological contexts [79, 124-128], it providing "a road to modified gravity theories" [46].

文献 [76] 将 (81) 式推广到二次引力情形，参见 (31) 式。值得注意的是，至少在聚焦不动点区域时，结果 (82) 独立于具体的 $f(R)$ 截断 [46, 124]。关系 (82) 在宇宙学背景中也有重要意义 [79, 124-128]，它提供了“修正引力理论的一条路径”[46]。

The RG improvement at the level of the field equations is somewhat simpler, as $\alpha = 0$ in Eq. (79). Assuming once again that the matter energy-momentum tensor $T_{\mu\nu}$ is separately conserved, the consistency condition reads

在场方程层面进行 RG 改进要更简单一些，因为此时 (79) 式中 $\alpha = 0$ 。再次假设物质能动张量 $T_{\mu\nu}$ 分别守恒，一致性条件为

$$\left(8\pi G_k' T_{\mu\nu} - \Lambda_k' g_{\mu\nu}\right) \nabla^\mu k(x) = 0, \tag{83}$$

and depends on the specific matter content of the system. This is to be contrasted with the improvement at the level of the action, where the contribution of the energy-momentum tensor cancels out. A particularly simple while important case is that of a perfect fluid. Indeed, perfect fluids play a role both in cosmology and in gravitational collapse models (see section "Gravitational Collapse and Improved Buchdahl Limit"). The energy-momentum tensor of a perfect fluid with energy density ρ and pressure $p = w\rho$ is $T_\mu^\nu = \mathrm{diag}(-\rho, p, p, p)$, and the corresponding consistency condition reduces to

且取决于系统的具体物质内容。这与作用量层面的改进形成对比，后者中能量动量张量的贡献会抵消。一个特别简单且重要的例子是理想流体。实际上，理想流体在宇宙学和引力坍缩模型中都有应用 (参见章节“引力坍缩与改进的布赫达限”)。能量密度为 ρ 、压强为 $p = w\rho$ 的理想流体，其能量动量张量为 $T_\mu^\nu = \mathrm{diag}(-\rho, p, p, p)$ ，对应的自洽条件可简化为

$$\frac{G_k'}{G_k}\left(\rho + \rho_\Lambda(k)\right) + \rho_\Lambda'(k) = 0, \tag{84}$$

where we have defined $\rho_\Lambda \equiv \Lambda_k/(8\pi G_k)$. Focusing once again on the fixed-point regime, the above relation suggests that [43,44]

其中我们定义了 $\rho_\Lambda \equiv \Lambda_k/(8\pi G_k)$。再次聚焦于不动点区域，上述关系表明 [43,44]

$$k^4 = \left(\frac{8\pi g_*}{\lambda_*}\right)\rho. \tag{85}$$

A cutoff identification of this type was used in [102-104] to study the RG-improved gravitational collapse of a massive star into a black hole, as well as in a cosmological context [129-132]. Indeed, in cosmology the metric is approximately that of a Friedmann-Lemaître-Robertson-Walker spacetime, the energy-density is related to the scale factor via $\rho(a) = \rho_0(a(t)/a_0)^{-3(1+w)}$ (at least assuming the standard conservation equations for $T_{\mu\nu}$, and a power-law behavior for $a(t)$) and $k \propto a(t)^{-\frac{3}{4}(1+w)}$. In turn, $a(t)$ can be written in terms of the cosmological time t or the Hubble constant $H(t)$. It follows that in the proximity of the fixed point, assuming no energy flow between the gravitational and matter sector, all cutoff identifications in terms of time t, scale factor $a(t)$, Hubble constant $H(t)$, and matter energy density $\rho(t)$ are physically equivalent (i.e., they produce the same results), as long as they have the correct power. On the other hand, in regimes where there is a substantial energy exchange between the gravitational and matter sectors, the Bianchi identities are automatically satisfied and thus do not allow to constrain the map $k(x)$.

这类截断识别已被文献 [102-104] 用于研究大质量恒星坍缩成黑洞的 RG 改进引力坍缩过程，也被应用于宇宙学背景中 [129-132]。实际上，宇宙学中的度规近似为弗里德曼-勒梅特-罗伯逊-沃尔克度规，能量密度通过 $\rho(a) = \rho_0(a(t)/a_0)^{-3(1+w)}$ 与标度因子关联 (至少在假设 $T_{\mu\nu}$ 满足标准守恒方程、且 $a(t)$ 呈幂律行为的条件下)，即 $k \propto a(t)^{-\frac{3}{4}(1+w)}$。相应地，$a(t)$ 可以用宇宙时间 t 或哈勃常数 $H(t)$ 表示。由此可得，在不动点附近，假设引力与物质 sector 之间没有能量流动，只要幂次正确，所有以时间 t、标度因子 $a(t)$、哈勃常数 $H(t)$ 和物质能量密度 $\rho(t)$ 表示的截断识别在物理上都是等价的 (即它们会给出相同的结果)。另一方面，在引力与物质 sector 之间存在显著能量交换的区域，比安基恒等式会自动满足，因此无法用来约束映射 $k(x)$。

Finally, as we shall see later, the relation (85) is also related to the decoupling mechanism: naïvely, as ρ also coincides with the Lagrangian density of a perfect fluid, it can act as a physical IR cutoff and thus contribute - together with other IR quantities - to the total decoupling scale k_{dec}. The difference between the implementation at the level of the action and at the level of the field equations is resolved when exploiting the decoupling mechanism: the decoupling condition imposes that the decoupling scale be a combination of both the curvature invariants (e.g., the Ricci scalar R in the Einstein-Hilbert truncation) and the matter energy density ρ [42].

最后，我们在后文会提到，式 (85) 还与退耦机制有关: 简单来说，由于 ρ 同时也等于理想流体的拉格朗日密度，它可以充当物理红外截断，和其他红外物理量一起对总退耦标度 k_{dec} 产生贡献。利用退耦机制可以解决作用量层面和场方程层面两种实现方式的差异: 退耦条件要求退耦标度是曲率不变量 (例如爱因斯坦-希尔伯特截断中的里奇标量 R) 和物质能量密度 ρ 的组合 [42]。

Self-Consistency and Iterative RG Improvement

自洽迭代 RG 改进

The determination of the RG-improved metric $g_{\mu\nu}^{qu}$ starting from the classical (typically Schwarzschild) background $\bar{g}_{\mu\nu}$ involves a scale identification $k = k(x)$ relating it to curvature invariants. Yet, all physical invariants, including the proper distance and the Kretschmann scalar in Eq. (23), are built on the classical background metric $\bar{g}_{\mu\nu} = g^{(0)}_{\mu\nu}$, which is singular and not reliable in the region $r \ll l_{Pl}$, where one needs to determine quantum gravity-induced modifications. The physical invariants constructed using the new, RG-improved metric $g^{qu}_{\mu\nu} = g^{(1)}_{\mu\nu}$ will generally differ from the classical ones, and therefore they would lead to a different functional form for $k(x)$. Moreover, at variance of the case of scalar QED, the functional expression of $k(x)$ - being built on the spacetime metric - depends explicitly on the coupling that is to be improved (i.e., the Newton coupling G_0). Overall, the improvement modifies the spacetime in a way that can impact the improvement itself and thus backreaction effects ought to be accounted for in the procedure. All these points suggest that the RG improvement in gravity should be implemented self-consistently.

从经典 (通常为史瓦西) 背景 $\bar{g}_{\mu\nu}$ 出发确定 RG 改进度规 $g^{qu}_{\mu\nu}$ ，需要通过标度识别 $k = k(x)$ 将其与曲率不变量联系起来。但所有物理不变量，包括式 (23) 中的固有距离和克雷奇曼标量，都构建在经典背景度规 $\bar{g}_{\mu\nu} = g^{(0)}_{\mu\nu}$ 之上；该度规是奇异的，在需要确定量子引力诱导修正的区域 $r \ll l_{Pl}$ 并不可靠。使用新的 RG 改进度规 $g^{qu}_{\mu\nu} = g^{(1)}_{\mu\nu}$ 构造的物理不变量通常会与经典不变量不同，因此会得到 $k(x)$ 的不同函数形式。此外，与标量量子电动力学的情况不同，构建在时空度规之上的 $k(x)$ 的函数表达式明确依赖于待改进的耦合 (即牛顿耦合 G_0)。总体而言，改进过程会以影响改进本身的方式改变时空，因此该过程需要考虑反作用效应。所有这些点都表明，引力中的 RG 改进应当以自洽的方式实现。

A possible approach to find a self-consistent quantum-corrected metric $g^{*}_{\mu\nu}$ is to define a sequence of RG improvements

寻找自洽量子修正度规 $g^{*}_{\mu\nu}$ 的一种可行方法是定义一系列 RG 改进过程

$$g^{cl}_{\mu\nu} = g^{(0)}_{\mu\nu} \to g^{(1)}_{\mu\nu} \to \ldots \to g^{(n)}_{\mu\nu} \to \ldots \to g^{(\infty)}_{\mu\nu} = g^{*}_{\mu\nu},$$

(86)

where the metric $g^{(n)}_{\mu\nu}$ at the step n and the corresponding effective Newton coupling $G_n(r)$ are defined via an IR cutoff which depends on the spacetime at the previous step $k_n\left[g^{(n-1)}_{\mu\nu}\right]$. If the procedure converges, it will lead to a self-consistent solution $g^{*}_{\mu\nu}$ which is a fixed point of the iterative procedure: a further RG improvement using the cutoff $k^{*}\left[g^{(*)}_{\mu\nu}\right]$ would lead to the same RG-improved metric $g^{*}_{\mu\nu}$.

其中第 n 步的度规 $g^{(n)}_{\mu\nu}$ 和对应的有效牛顿耦合 $G_n(r)$ 由红外截断定义，该截断依赖于前一步 $k_n\left[g^{(n-1)}_{\mu\nu}\right]$ 的时空。如果该过程收敛，最终将得到自洽解 $g^{*}_{\mu\nu}$ ，它是迭代过程的不动点: 使用截断 $k^{*}\left[g^{(*)}_{\mu\nu}\right]$ 进行进一步 RG 改进仍会得到相同的 RG 改进度规 $g^{*}_{\mu\nu}$ 。

This iterative procedure was first devised and applied in [133]. However, it is worth mentioning that a precursor of these ideas already appeared in [86]. Indeed, [86] discussed RG-improved Kerr-(A)dS and Schwarzschild-(A)dS space-times, using a scale identification based on a partially improved Kretschmann scalar, where the ”partial” refers to the omission of the derivatives of the effective Newton coupling $G(r)$ in the expression of the Kretschmann scalar. Using this partially, self-consistent choice, the singularity is not resolved, although it is weaker and some geodesics do not hit it. Other features of the resulting partially self-consistent black holes, such as the horizons and the evaporation process, resemble those of Bonanno-Reuter black holes.

该迭代过程最早由文献 [133] 设计并应用。但值得一提的是，这些思想的雏形早已出现在文献 [86] 中。事实上，文献 [86] 讨论了 RG 改进的克尔-(A)dS 和史瓦西-(A)dS 时空，采用了基于部分改进克雷奇曼标量的标度识别，这里的“部分”指在克雷奇曼标量的表达式中省略了有效牛顿耦合 $G(r)$ 的导数项。采用这种部分自洽的选择后，奇点并未被消除，只是强度减弱，且有部分测地线不会撞上奇点。由此得到的部分自洽黑洞的其他特征，例如视界和蒸发过程，与博南诺-罗伊特黑洞的特征相似。

In [133] the self-consistent RG-improvement procedure was implemented iteratively. For $n > 1$ the metric is determined by the replacement rule

文献 [133] 以迭代方式实现了自洽 RG 改进过程。对于 $n > 1$，度规由替换规则确定

$$G_{(n)} \to G_{(n+1)}(r) = \frac{G_0}{1 + g_*^{-1} G_0 k_{(n+1)}^2(r)}, \tag{87}$$

where the cutoff function $k_{(n+1)}$, independent of its functional form (e.g., based on the improved Ricci or Kretschmann scalars), can be written as a functional of the energy-density $\rho_n(r)$ (cf. Eq. (61)) induced by the spatial variation of $G_{(n)}(r)$ in the previous step

其中截断函数 $k_{(n+1)}$ 不依赖于自身的函数形式 (例如基于改进里奇标量或改进克雷奇曼标量)，可以写为前一步 $G_{(n)}(r)$ 空间变化诱导的能量密度 $\rho_n(r)$ 的泛函 (参见式 (61))

$$k_{(n+1)}^2(r) \equiv \mathcal{K}\left[\rho_{(n)}(r)\right]. \tag{88}$$

Consequently, the sequence of RG improvements is encoded in a recursive relation

因此，一系列 RG 改进可以用递推关系表示

$$G_{(n+1)}(r) = \frac{G_0}{1 + g_*^{-1} G_0 \mathcal{K}\left[G'_{(n)}(r)\right]}, \tag{89}$$

whose fixed point - reached in the limit $n \to \infty$ under the assumption of convergence - is the solution to the following differential equation:

在收敛假设下，该递推关系在极限 $n \to \infty$ 达到的不动点是以下微分方程的解:

$$\mathcal{K}\left[G'_\infty(r)\right] = g_* \frac{G_0 - G_\infty(r)}{G_0 G_\infty}. \tag{90}$$

Note that above and in the following, we shall adopt the more compact notation G_∞ in place of $G_{(\infty)}$ for shortness. Once a specific cutoff identification $\mathcal{K}$ has been specified, a solution can be found. While the topic of uniquely identifying $k(x)$ is the subject of section ”Effective Solutions from the Decoupling Mechanism,” it is instructive to see how the iterative RG improvement works for a specific choice of $\mathcal{K}$. Motivated by the form of the improved Ricci and Kretschmann scalars [133], where one can see that the quantity $G_0\rho$ acts as an effective IR cutoff,[133] set $k^2 \equiv \mathcal{K}[\rho] = G_0\rho$. The resulting fixed-point equation is the following first-order differential equation:

请注意，为简便起见，在上文及下文我们都将采用更紧凑的记号 G_∞ ，而非 $G_{(\infty)}$ 。一旦确定了特定的截断识别 $\mathcal{K}$ ，即可求得解。尽管唯一确定 $k(x)$ 这一主题是“退耦机制的有效解”一节的讨论主题，了解迭代重整化群改进对特定选择的 $\mathcal{K}$ 如何运作仍具有启发意义。受改进后的里奇标量和克雷奇曼标量形式 [133] 的启发——其中可以看出物理量 $G_0\rho$ 充当有效红外截断 [133]——我们设定 $k^2 \equiv \mathcal{K}[\rho] = G_0\rho$ 。所得不动点方程为如下一阶微分方程:

$$G_\infty(r) = \frac{4\pi r^2 G_0 G_\infty(r)}{4\pi r^2 G_\infty(r) + g_*^{-1} G_0^2 m G'_\infty(r)}, \tag{91}$$

which can be solved exactly and leads to a regular spacetime metric that was put forth two decades earlier by Dymnikova [134] (see Fig.9). Interestingly, in the classical limit (that can be obtained by pushing the RG fixed point to infinity, $g_* \rightarrow \infty$), one recovers the Schwarzschild metric, whereas the case of asymptotic freedom $g_* \rightarrow 0$ would require the effective Newton coupling to be everywhere zero.

该方程可精确求解，得到了 Dymnikova 早在二十年前就提出的正则时空度规 (参见图 9)。有趣的是，在经典极限下 (可通过将 RG 不动点推至无穷远得到，即 $g_* \rightarrow \infty$)，我们可以重新得到史瓦西度规，而渐近自由情形 $g_* \rightarrow 0$ 则要求有效牛顿耦合处处为零。

Coordinate Dependence and Invariant RG Improvement

坐标依赖性与不变量重整化群改进

The RG improvement at the level of the solutions implements the replacement $G_0 \rightarrow G_k$ and the subsequent identification $k \mapsto k(x)$ in the metric coefficients. As argued in [135], a key question is whether this procedure is coordinate independent, i.e., whether the physical properties of the resulting RG-improved spacetime (for instance, its curvature invariants) depend on the initial choice of coordinates.

解层面的重整化群改进是在度规系数中完成替换 $G_0 \rightarrow G_k$ ，随后做标识 $k \mapsto k(x)$ 。正如文献 [135] 所述，核心问题是该过程是否坐标无关，即得到的重整化群改进时空的物理性质 (例如其曲率不变量) 是否依赖于初始坐标选择。

While the RG improvement at the level of the action ought to be coordinate independent due to the Lagrangian being a scalar density, a metric $g_{\mu\nu}$ transforms as a $(0,2)$ -tensor under coordinate transformations, implying that the RG improvement at the level of the metric is generally coordinate dependent [135]. In the following we review a possible solution to the problem of coordinate dependence - otherwise dubbed ”invariant RG improvement” [135].

尽管作用量层面的重整化群改进因拉格朗日量是标量密度理应坐标无关，但度规 $g_{\mu\nu}$ 在坐标变换下按 $(0,2)$ 张量变换，这意味着度规层面的重整化群改进一般是坐标依赖的 [135]。下文我们将综述解决坐标依赖性问题的一种可行方案——也就是所谓的“不变量重整化群改进”[135]。

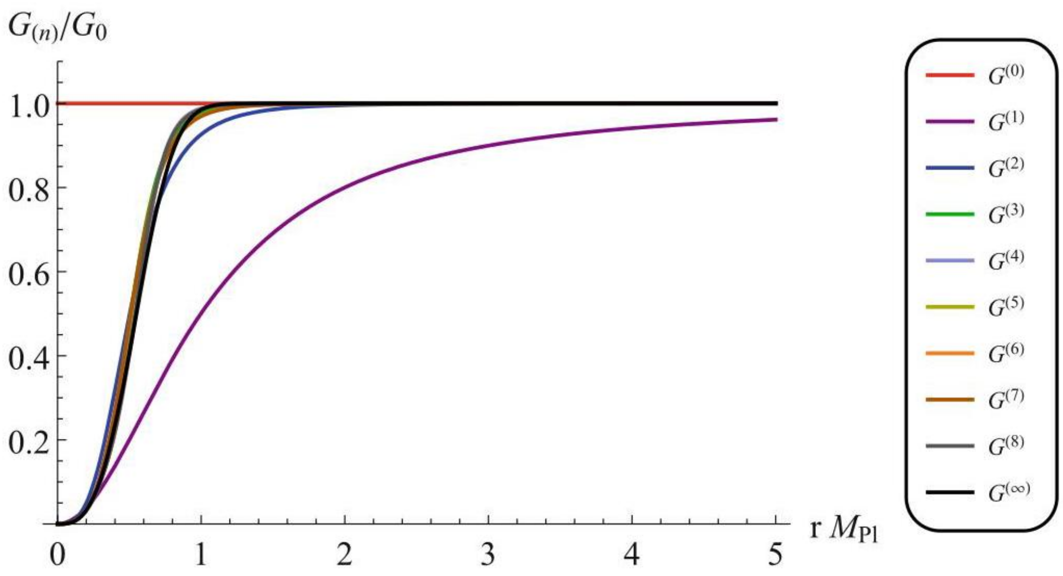


Fig. 9 Convergence of the effective Newton coupling $G_n(r)$ (shown in Planck units) in the iterative RG improvement procedure. At the step $n = 0, G_{n=0} = G_0$. In the first step, the function $G_1(r)$ interpolates between its IR value G_0 and zero in the UV. Successive steps display a fast convergence toward the fixed-point solution $G_\infty(r)$

图 9 迭代重整化群改进过程中有效牛顿耦合 $G_n(r)$ 的收敛性 (以普朗克单位表示)。迭代到第 $n = 0, G_{n=0} = G_0$ 步。第一步中，函数 $G_1(r)$ 在其红外值 G_0 和紫外零值之间插值。后续迭代快速向不动点解 $G_\infty(r)$ 收敛。

To tackle the question of coordinate dependence, [135] exploits the characterization of spacetimes of the Petrov and Segre types [136] via a complete set of curvature invariants named Zakhary-McIntosh (ZM) invariants [137-139]. As these quantities transform as scalars under diffeomorphisms, they can be used to explicitly show that the RG improvement at the level of the metric is coordinate dependent. While we will not go through the details of the proof, which can be found in [135], the intuitive cause of such a coordinate dependence is simply the replacement of a scalar quantity ($G_0 \to G[k(r)]$) in an object (the metric coefficients) that does not transform as a scalar under a diffeomorphism transformation. This intuitive understanding also points to a solution to the problem of coordinate dependence: implementing the RG improvement at the level of the curvature invariants. Specifically, the invariant RG improvement devised in [135] consists in performing the replacement $G_0 \to G_k$ in (one of) the functionally independent ZM invariants.

为解决坐标依赖性问题，文献 [135] 利用完整的曲率不变量集合扎哈里-麦金托什 (ZM) 不变量 [137-139] 表征佩特罗夫型和塞格雷型时空 [136]。由于这些量在微分同胚下按标量变换，它们可被用于直接证明度规层面的重整化群改进是坐标依赖的。我们不会展开证明细节，相关内容可参见文献 [135]，这种坐标依赖性的直观成因很简单: 在微分同胚变换下不按标量变换的对象 (即度规系数) 中替换一个标量 ($G_0 \to G[k(r)]$)。这种直观理解也指出了坐标依赖性问题的解决方向: 在曲率不变量层面实施重整化群改进。具体而言，文献 [135] 设计的不变量重整化群改进就是在 (一个) 函数独立的 ZM 不变量中完成替换 $G_0 \to G_k$。

In order to make this method more concrete, let us focus on the case of spherically symmetric black holes and highlight the practical differences between the metric and invariant RG improvements. The metric is in this case diagonal, and, assuming $g_{rr} = g_{tt}^{-1}$, it is given by Eq. (5). A Schwarzschild black hole is characterized by the classical lapse function

为了让该方法更具体，我们以球对称黑洞为例，说明度规重整化群改进与不变量重整化群改进的实际差异。该情形下度规是对角的，且在假设 $g_{rr} = g_{tt}^{-1}$ 下，其形式由式 (5) 给出。史瓦西黑洞由经典时延函数表征为

$$f_{cl}\left(r\right) = 1 - \frac{2mG_0}{r}, \tag{92}$$

and by only one independent ZM invariant

且仅存在一个独立的 ZM 不变量

$$\left(\frac{\mathcal{K}_1^{cl}}{48}\right)^3 = \left(\frac{\mathcal{K}_3^{cl}}{96}\right)^2 = \left(\frac{G_0 M}{r^3}\right)^6, \; \mathcal{K}_{i\neq 1,3}^{cl} = 0. \tag{93}$$

At this point the standard RG improvement at the level of the metric follows the rule

至此，度规层面的标准重整化群改进遵循如下规则

$$f_{cl}\left(r\right) \underset{\text{Metric RG impr.}}{\rightarrow} f_{qu} = 1 - \frac{2mG\left[k\left(r\right)\right]}{r}, \tag{94}$$

for some choice of $k = k\left(r\right)$, while the invariant RG improvement is defined by

对 $k = k\left(r\right)$ 的某种选择，而不变量重整化群改进定义为

$$\mathcal{K}_1^{cl}\left(r\right) \underset{\text{Invariant RG impr.}}{\rightarrow} \mathcal{K}_1^{qu} = 48\left(\frac{mG\left[k\left(r\right)\right]}{r^3}\right)^2. \tag{95}$$

Within this scheme, the new, RG-improved lapse function is defined by the differential equation

在该框架下，新的经重整化群改进的时延函数由如下微分方程定义

$$\mathcal{K}_1 = \frac{\left(r^2 f_{qu}'' - 2r f_{qu}' + 2f_{qu} - 2\right)^2}{3r^4} \equiv \mathcal{K}_1^{qu}. \tag{96}$$

The result at this point depends on the cutoff function $k\left(r\right)$. Identifying, for instance, $k^4 \sim \mathcal{K}_1 \equiv K$ yields the effective RG-improved metric (see Fig. 10)

此时结果依赖于截断函数 $k\left(r\right)$ 。例如，取标识 $k^4 \sim \mathcal{K}_1 \equiv K$ 即可得到有效重整化群改进度规 (参见图 10)

$$f_{qu}\left(r\right) = 1 - \frac{2r\ell}{g_*^{-1} l_{Pl}^2}\left[\sqrt{3}\pi + 2\sqrt{3}\arctan\left(\frac{1 - 2r/\ell}{\sqrt{3}}\right)\right.$$
$$\left. + \log\left(\ell^2 - r\ell + r^2\right) - 2\log\left(\ell + r\right)\right] \tag{97}$$

$$+\frac{4r^2}{g_*^{-1}l_{Pl}^2}\left[\log\left(\ell^3+r^3\right)-3\log\left(r\right)\right],$$

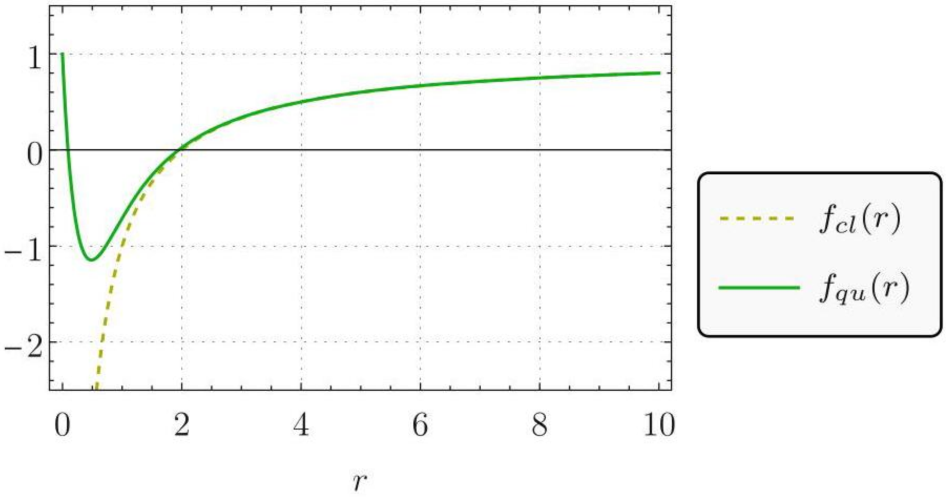


Fig. 10 Comparison of the classical lapse function (dashed yellow line) with the one in Eq. (97) (solid green line), which was first computed in [135] via the invariant RG improvement. The figure refers to the case of a black hole of Planckian mass, $m = m_{Pl}$, and we set $g_* = 1$. The critical value to merge and remove the horizons is in this case sub-Planckian, $m_c < m_{Pl}$.

图 10 经典时延函数 (黄色虚线) 与式 (97) 给出的时延函数 (绿色实线) 的对比，后者由文献 [135] 最早通过不变量重整化群改进计算得到。本图对应普朗克质量黑洞的情形，$m = m_{Pl}$ ，我们取 $g_* = 1$ 。该情形下视界融合并消除的临界值为亚普朗克量级，$m_c < m_{Pl}$ 。

with $\ell = \sqrt[3]{g_*^{-1}l_{Pl}^4 m}$. An asymptotic expansion of this lapse function shows that this metric reproduces the Schwarzschild spacetime for $r \to \infty$, together with additional corrections. In the opposite limit, $r \to 0$, where deviations due to quantum gravity are stronger, the lapse function depends linearly on the radial coordinate. Quantum gravity effects thus make the singularity weaker, but do not resolve it. A possible reason, as argued in [135], is that the invariant RG improvement ought to be combined with the iterative procedure described in the previous section; this combination would allow to determine an effective self-consistent metric via a coordinate-independent procedure. On top of this, a clear recipe to fix the functional relation $k(x)$ uniquely would make the RG improvement procedure overall more consistent and rigorous. The latter issue will be the focus of the next subsection.

结合 $\ell = \sqrt[3]{g_*^{-1}l_{Pl}^4 m}$ 。对该时移函数做渐近展开后可知，当 $r \to \infty$ 时，该度量会重现史瓦西时空，同时带有额外修正。在相反极限即 $r \to 0$ 下，量子引力带来的偏差更强，此时时移函数与径向坐标呈线性相关。因此量子引力效应会弱化奇点，但无法消除奇点。正如文献 [135] 所述，一个可能的原因是: 不变性 RG 改进应当结合前一节介绍的迭代过程；通过这种坐标无关的步骤，二者结合就能确定有效自洽度量。除此之外，一套明确的方法来唯一确定函数关系 $k(x)$ ，能让整个 RG 改进过程更加自洽、严谨。这一问题将是下一小节的讨论核心。

Effective Solutions from the Decoupling Mechanism

退耦机制的有效解

This subsection focuses on the findings in [42], which for the first time exploited the decoupling mechanism (cf. section "RG Improvement: Key Idea") to uniquely determine the functional relation $k(x)$ grounded on RG considerations. The resulting framework was then applied, in combination with the iterative RG improvement (cf. section "Self-Consistency and Iterative RG Improvement") to investigate the dynamics of self-consistent quantum-corrected black holes from formation to evaporation, within the same VKP collapse model used in section "Gravitational Collapse and Improved Buchdahl Limit."

本小节重点介绍文献 [42] 的研究成果，该研究首次利用退耦机制 (参见“RG 改进: 核心思想”小节)，基于重整化群考虑唯一确定了函数关系 $k(x)$。随后将得到的框架结合迭代 RG 改进 (参见“自洽性与迭代 RG 改进”小节)，用于研究在“引力坍缩与改进的布赫达尔极限”小节所使用的同一 VKP 坍缩模型中，自洽量子修正黑洞从形成到蒸发的动力学过程。

The first step to exploit the decoupling mechanism [41] is to find a well-defined mathematical expression for the decoupling condition. Its determination, as briefly discussed in section "RG Improvement: Key Idea," could yield a shortcut from a truncated version of the EAA Γ_k to the effective action Γ_0 (or its solutions, to some extent), typically within a larger truncation than the original one. Inasmuch as the decoupling occurs when physical IR scales in the effective action (if any) overcome the artificial regulator $\mathcal{R}_k \sim k^2$, the decoupling condition can be mathematically characterized at the level of the inverse propagator. The latter has the following schematic structure:

利用退耦机制 [41] 的第一步是为退耦条件给出一个定义明确的数学表达式。正如“RG 改进: 核心思想”小节简要讨论的，确定该条件可以提供一条捷径，从截断形式的有效平均作用量 Γ_k 得到有效作用量 Γ_0 (或在一定程度上得到其解)，通常该过程的截断范围比原始截断更大。当有效作用量中的物理红外尺度 (若存在) 超过人工调节因子 $\mathcal{R}_k \sim k^2$ 时，退耦就会发生，因此退耦条件可以在逆传播子层面通过数学表征。逆传播子具有如下概型结构:

$$\Gamma_k^{(2)} + \mathcal{R}_k = c\left(p^2 + A_k[\Phi] + \widetilde{\mathcal{R}}_k\right), \tag{98}$$

where c is a constant, Φ is the set of fields considered, and $A_k[\Phi] \equiv \Gamma_k^{(2)}/c - p^2$ and $\mathcal{R}_k \equiv c\widetilde{\mathcal{R}}_k$. The decoupling condition thus reads [42]

其中 c 是常数，Φ 是所考虑的场集合，$A_k[\Phi] \equiv \Gamma_k^{(2)}/c - p^2$ 和 $\mathcal{R}_k \equiv c\widetilde{\mathcal{R}}_k$。因此退耦条件可写为 [42]

$$\widetilde{\mathcal{R}}_{k_{\rm dec}} \approx A_{k_{\rm dec}}[\Phi] \tag{99}$$

and provides an implicit way to uniquely identify k as the decoupling scale $k_{\rm dec}$. Notably, even in the presence of multiple physical IR scales (e.g., masses, curvature invariants, and matter energy density), the decoupling scale is given by a precise combination of all of them. This is not surprising, as the dynamics of a system typically depends on all its components; similarly, all physical scales should contribute - if perhaps with different weights - to the determination of the sought-after decoupling scale.

并为将 k 唯一确定为退耦尺度 $k_{\rm dec}$ 提供了隐式方法。值得注意的是，即使存在多个物理红外尺度(例如质量、曲率不变量和物质能量密度)，退耦尺度也可由所有这些尺度的精确组合给出。这并不奇怪，因为系统的动力学通常依赖于它的所有组分；同理，所有物理尺度都应当对目标退耦尺度的确定做出贡献——只是权重可能不同。

It is pedagogical to see explicitly how this works in massless scalar QED [140]. For this case, $\Gamma_k^{(2)} \approx p^2 + \lambda_k \phi^2 + \widetilde{\mathcal{R}}_k + \ldots$, with $\lambda_k \approx \log k$ being the coupling in front of the ϕ^4 interaction term. Choosing a mass-type regulator, $\widetilde{\mathcal{R}}_k \sim k^2$, the decoupling condition yields - to leading-order - the Coleman-Weinberg potential $\phi^4 \log \phi$.

我们可以通过无质量标量量子电动力学 [140] 直观了解这一机制的具体运作。在该情形下，$\Gamma_k^{(2)} \approx p^2 + \lambda_k \phi^2 + \widetilde{\mathcal{R}}_k + \ldots$ ，其中 $\lambda_k \approx \log k$ 是 ϕ^4 相互作用项前的耦合。选取质量型调节因子 $\widetilde{\mathcal{R}}_k \sim k^2$ ，退耦条件在领头阶给出了科尔曼-温伯格势 $\phi^4 \log \phi$ 。

Let us now turn on gravity and exploit the decoupling mechanism to study the quantum-corrected gravitational collapse in a VKP model. Within the Einstein-Hilbert truncation, the EAA reads

现在我们引入引力，利用退耦机制研究 VKP 模型中量子修正的引力坍缩。在爱因斯坦-希尔伯特截断下，有效平均作用量为

$$\Gamma_k = \int \mathrm{d}^d x \sqrt{g} \left(\frac{1}{16\pi G_k} \left(2\Lambda_k - R\right) + \mathcal{L}_m \right), \tag{100}$$

where G_k and Λ_k are the RG scale-dependent Newton coupling and cosmological constant, d is the number of spacetime dimensions, and $\mathcal{L}_m$ is the Lagrangian of a pressureless perfect fluid (radiation) [141]

其中 G_k 和 Λ_k 是依赖于 RG 尺度的牛顿耦合和宇宙学常数，d 是时空维度，$\mathcal{L}_m$ 是无压强理想流体(辐射) 的拉格朗日量 [141]

$$\mathcal{L}_m = \mu\,(r, v)\,, \tag{101}$$

as required by the VKP model. Complementing the gravitational EAA with a harmonic gauge fixing and the standard Faddeev-Popov ghost action, setting $d = 4$, and restricting to a mass-type regulator, the decoupling condition reads

这是 VKP 模型的要求。给引力有效平均作用量补充谐和规范固定和标准法捷耶夫-波波夫鬼作用量，令 $d = 4$ ，并限定使用质量型调节因子，退耦条件可写为

$$k_{\rm dec}^2 \equiv G_{k_{\rm dec}} \mu + \frac{2}{3} R, \tag{102}$$

where $G_{k\,{\rm dec}}$ is given by Eq. (9), as usual, with $k \equiv k_{\rm dec}$. One can straightforwardly see that replacing this condition in the EAA introduces higher-derivative operators in the effective action, as expected. We note at this point that R can always be written in terms of the gravity-induced energy density and pressure (61); in turn μ, ρ , and p depend on the effective Newton coupling $G\,(r, v)$ and its partial derivatives. Therefore, one can straightforwardly apply the differential equation defining the fixed point of the iterative RG improvement - Eq.

(90) - with the cutoff identification $\mathcal{K}$ dictated by the decoupling condition (102). The differential equation determining the self-consistent effective Newton coupling is [42]

其中 $G_{k_{\text{dec}}}$ 由式 (9) 给出，和通常情况一样，满足 $k \equiv k_{\text{dec}}$ 。不难发现，将该条件代入有效平均作用量后，正如预期会在有效作用量中引入高阶导数算符。此处我们注意到，R 总能以引力诱导能量密度和压强 (61) 的形式写出；相应地，μ, ρ 和 p 依赖于有效牛顿耦合 $G(r, v)$ 及其偏导数。因此，我们可以直接对由迭代 RG 改进不动点定义的微分方程——即式 (90)——应用退耦条件 (102) 给出的截断标识 $\mathcal{K}$ 。决定自洽有效牛顿耦合的微分方程为 [42]

$$G_\infty = \frac{G_0}{1 + g_*^{-1} G_0 G_\infty \left(\mu_\infty + \frac{2}{3} 16\pi \left(\rho_\infty - p_\infty\right)\right)}, \tag{103}$$

where μ_∞, ρ_∞ , and p_∞ are given by

其中 μ_∞, ρ_∞ 和 p_∞ 由下式给出

$$\mu_\infty = \frac{\dot{m}(v)}{4\pi G_\infty r^2} + \frac{m(v)\,\dot{G}_\infty}{4\pi G_\infty r^2},\ \rho_\infty = \frac{m(v)\,G'_\infty}{4\pi G_\infty r^2},\ p_\infty = -\frac{m(v)\,G''_\infty}{8\pi G_\infty r^2},$$

(104)

and, as usual, primes and dots denote spatial and advanced time derivatives, respectively. Solutions to this differential equation for a given collapse model, specified by the mass function $m(v)$, correspond to quantum-corrected dynamical spacetimes with lapse function

和通常情况一样，撇号和点分别表示空间导数和超前时间导数。对于由质量函数 $m(v)$ 指定的给定坍缩模型，该微分方程的解对应带 lapse 函数的量子修正动态时空

$$f_{qu}(r, v) \equiv f_\infty(r, v) = 1 - \frac{2m(v)\,G_\infty(r, v)}{r}, \tag{105}$$

describing the spacetime dynamics from formation to evaporation. In particular, [42] studied in detail solutions to the defining differential equation (103) for a VKP model, both analytically, in different limiting regimes, and numerically. The full numerical solution is displayed in Fig. 11. and is derived by imposing that the observed Newton coupling is recovered at early times (before the collapse starts, for any r) and at large distances, $G_\infty(r, v_0) = G_\infty(r_{\max}, v) = G_0$. The remaining boundary condition is set by determining the asymptotic behavior of the solution analytically, [42]. By construction, the effective Newton coupling reproduces the observed value of Newton constant at early times and at large distances. During the collapse the effective Newton coupling decays as $\sim v^{-1}$. At the end of the collapse, the effective Newton coupling converges to a function that connects smoothly the IR regime (large radii), where $G_\infty(r) \to G_0$, and the UV fixed-point scaling $(r \to 0)$, where the effective Newton coupling vanishes. While all these features were expected from previous studies, quantum-corrected black holes stemming from the decoupling mechanism feature an additional striking feature reminiscent of higher-derivative operators with specific nonlocal (exponential) form factors [142]: damped oscillations along the radial direction. The presence of non-trivial form factors is crucial to explain the oscillations. Indeed, black holes in quadratic gravity lead to free oscillations provided that a specific sign for the Weyl square term C^2 in the action is used [143-145], while the damping can be obtained from the presence of exponential form factors [142]. This is in line with the expectation that the decoupling mechanism ought to grant access to higher-derivative operators that were not initially considered in the original truncation for the EAA - the Einstein-Hilbert one in the case of [42].

描述从形成到蒸发的时空动力学。特别地，文献 [42] 针对 VKP 模型，在不同极限区域分别通过解析方法和数值方法详细研究了定义方程 (103) 的解。完整数值解如图 11 所示，求解时施加了如下条件: 在早期 (坍缩开始前，对任意 r) 和大距离 $G_\infty(r, v_0) = G_\infty(r_{\max}, v) = G_0$ 处，能够恢复观测到的牛顿耦合。剩余边界条件由解析求解的渐近行为确定 [42]。根据构造，有效牛顿耦合在早期和大距离处都会重现牛顿常数的观测值。坍缩过程中，有效牛顿耦合按 $\sim v^{-1}$ 衰减。坍缩结束时，有效牛顿耦合收敛到一个函数，该函数平滑连接红外区域 (大半径，此处满足 $G_\infty(r) \to G_0$) 和紫外不动点标度区 ($r \to 0$) , 有效牛顿耦合在后者中趋于零。尽管所有这些特征都和之前研究的预期一致，由退耦机制得到的量子修正黑洞还存在一个额外的显著特征——沿径向的阻尼振荡，这让人联想到具有特定非局域 (指数) 形状因子的高阶导数算符 [142]。非平庸形状因子的存在是解释振荡的关键。实际上，只要作用量中 Weyl 平方项 C^2 取特定符号，二次引力中的黑洞就会产生自由振荡 [143-145]，而阻尼可由指数形状因子的存在产生 [142]。这符合预期: 退耦机制应当能够得到初始有效平均作用量截断 (文献 [42] 中为爱因斯坦-希尔伯特截断) 中没有考虑的高阶导数算符。

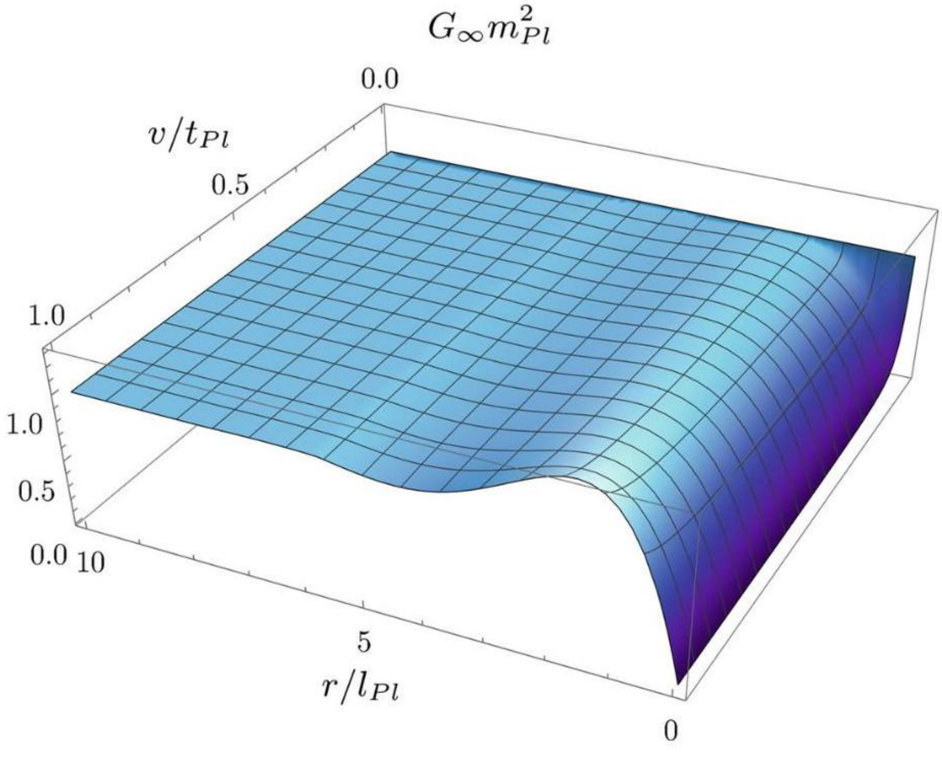


Fig. 11 Numerical solution to the partial differential equation (103) self-consistently defining the effective Newton coupling associated with a dynamical quantum-corrected spacetime stemming from the decoupling mechanism [42]. In order to solve Eq. (103), [42] used the boundary and initial conditions $G_\infty(r, v_0) = G_\infty(r_{\max}, v) = G_0$. A final boundary condition is imposed by requiring the solution to match the solution close to the would-be singularity, as this can be studied analytically [42]. By construction, the effective Newton coupling reproduces the observed value of Newton constant at early times and at large distances. For $v > 0$ and during the entire duration of the collapse, the advanced time dependence of the effective Newton coupling is $\sim v^{-1}$. Once the infusion of radiation from the nearby massive star is over, the collapse ends and the spacetime is described by an effective Newton coupling which is monotonic with respect to the radial coordinate and

smoothly connects the limiting values G_0 , in the IR, and 0, in the UV. In addition, this dynamical black hole spacetime is characterized by damped oscillations along the radial direction, reminiscent of higher-derivative operators with specific nonlocal form factors [142]

> 图 11: 偏微分方程 (103) 的数值解，该方程自洽地定义了与退耦机制产生的动力学量子修正时空相关的有效牛顿耦合 [42]。为求解方程 (103)，文献 [42] 采用了边界条件和初始条件 $G_\infty(r, v_0) = G_\infty(r_{\max}, v) = G_0$ 。通过要求解在拟奇点附近与解析研究的结果匹配，施加了最终边界条件 [42]。根据构造，有效牛顿耦合在早期和大距离处还原为观测到的牛顿常数值。对于 $v > 0$ ，在整个坍缩过程中，有效牛顿耦合的超前时间依赖为 $\sim v^{-1}$ 。一旦附近大质量恒星的辐射注入结束，坍缩停止，此时时空由有效牛顿耦合描述，该耦合关于径向坐标单调，并平滑连接红外极限 G_0 和紫外极限 0。此外，该动态黑洞时空的特征是沿径向存在阻尼振荡，让人联想到具有特定非定域形状因子的高阶导数算符 [142]

One may speculate that exponential form factors would lead to exponential lapse functions; this expectation can be partially verified by studying the static limit of the dynamical solution, corresponding to the static configuration $m(v) \to m$ at the end of the collapse. This can be done analytically, within three complementary approximations: the short-distance regime close to the classical singularity, the large-distance limit, and an interpolation between the two that neglects the damped oscillations. The first limiting regime is reached by approximating $G_k \sim g_* k^{-2}$ in the derivation of Eq. (103). This is tantamount to neglecting the 1 in the denominator of Eq. (103). The resulting equation

> 我们可以推测指数形状因子会导出指数推移函数；这一预期可以通过研究动态解的静态极限 (对应坍缩结束后的静态构型 $m(v) \to m$) 得到部分验证。这可以在三种互补近似下通过解析完成: 靠近经典奇点的短距离区域、大距离极限，以及忽略阻尼振荡的两者间插值。第一个极限区域可通过在推导方程 (103) 时近似 $G_k \sim g_* k^{-2}$ 得到，这等价于忽略方程 (103) 分母中的 1。所得方程

$$G_0 g_*^{-1} m \left(4rG_\infty'' + 8G_\infty'\right) G_\infty - 3G_0 r^2 = 0, \tag{106}$$

can be solved by parameterizing $G_\infty(r) \sim Cr^n$ close to $r = 0$. With this strategy one finds the leading-order behavior of $G(r)$ close to the classical singularity:

> 可通过参数化 $G_\infty(r) \sim Cr^n$ 在 $r = 0$ 附近求解。利用该方法可以得到 $G(r)$ 在经典奇点附近的领头阶行为:

$$G_\infty(r) = \frac{1}{\sqrt{5g_*^{-1}m}} r^{3/2}. \tag{107}$$

This scaling (dotted line in the right panel of Fig. 12) is sufficient to make $G_\infty(r)$ vanish in the UV, but the decay with the radial coordinate r is not fast enough to resolve the singularity (cf. section ”The Role of the Cosmological Constant”). This is nonetheless expected from general considerations from the gravitational collapse (cf. section ”Gravitational Collapse and Improved Buchdahl Limit”). Next one could consider the opposite limit, $r \to \infty$, where the differential equation (103) reduces to

该标度行为 (图 12 右面板中的虚线) 足以让 $G_\infty(r)$ 在紫外趋于零，但随径向坐标 r 的衰减速度不足以解决奇点问题 (参见“宇宙学常数的作用”一节)。这一结果符合引力坍塌一般分析的预期 (参见“引力坍缩与改进的布赫达希极限”一节)。接下来我们考虑相反极限 $r \to \infty$，此时微分方程 (103) 简化为

$$\left(G_0 g_*^{-1} m\left(4rG_\infty'' + 8G_\infty'\right) + 3r^2\right) G_\infty - 3G_0 r^2 = 0. \tag{108}$$

In order to solve this equation, one can make the ansatz [42]

为求解该方程，我们可以做如下假设 [42]

$$G_\infty(r) = G_0\left(1 - \frac{F(r)}{r}\right), \tag{109}$$

with $|F(r)/r| \ll 1$ asymptotically and $|F(r)/r| \to 0$ as $r \to \infty$. Inserting the ansatz (109) into the differential equation (108) and solving with respect to the function $F(r)$ yields the solution

其中 $|F(r)/r| \ll 1$ 渐近成立，且 $|F(r)/r| \to 0$ 满足 $r \to \infty$。将假设 (109) 代入微分方程 (108)，对函数 $F(r)$ 求解得到解:

$$F(r) = \mathfrak{Re}\left[c_1 \,\mathrm{Ai}\left(a(m, g_*)\, r\right) + c_2 \,\mathrm{Bi}\left(a(m, g_*)\, r\right)\right], \tag{110}$$

where $a(m, g_*) = 2^{-2/3} 3^{1/3}\left(-G_0^2 m g_*^{-1}\right)^{-1/3}$ and $c_i \propto 1/m$ on dimensional grounds. The corresponding effective Newton coupling is displayed in Fig. 12. Finally, neglecting the damped oscillations, one can determine an interpolating function between the power-law scaling (107) at small radii and the Newton constant at large distances. This function reads

其中基于量纲分析可得 $a(m, g_*) = 2^{-2/3} 3^{1/3}\left(-G_0^2 m g_*^{-1}\right)^{-1/3}$ 和 $c_i \propto 1/m$。对应的有效牛顿耦合如图 12 所示。最后，忽略阻尼振荡，我们可以确定一个插值函数，连接小半径处的幂律标度 (107) 和大距离处的牛顿常数。该函数为

$$G_\infty(r) = G_0\left(1 - e^{-\frac{r^{3/2}}{\sqrt{5g_*^{-1} 2mG_0/2}\, l_{Pl}}}\right), \tag{111}$$

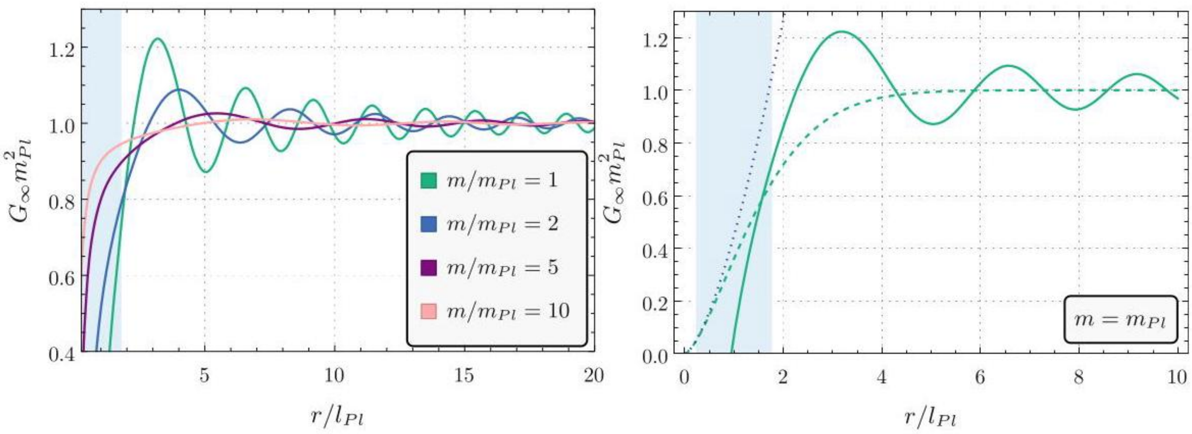


Fig. 12 Static limit of the dynamical effective Newton coupling at the end of the collapse. The figure on the left depicts the static solution at large distances, for different values of the black hole mass (corresponding to differently colored lines). It highlights that oscillations are present also in the static limit, with an amplitude that depends on the black hole mass. The figure on the right refers to Planckian black holes only and compares three approximations: the $\sim r^{3/2}$ scaling close to the singularity (dotted line, Eq. (107)), the large-distance solution described by Airy functions (solid line, Eq. (110)), and the interpolating function which connects these two regimes but neglects oscillations (dashed line, Eq. (111)). The blue region denotes the transition zone between the small and large radii limit: in this region the two limiting solutions (dotted and solid lines) cannot be trusted [42]

图 12 坍缩结束后动力学有效牛顿耦合的静态极限。左图描绘了不同黑洞质量 (对应不同颜色线条) 下大距离处的静态解。图中显示，静态极限下也存在振荡，振荡振幅取决于黑洞质量。右图仅针对普朗克黑洞，对比了三种近似: 奇点附近的 $\sim r^{3/2}$ 标度 (虚线，式 (107))、由艾里函数描述的大距离解 (实线，式 (110))，以及连接这两个区域但忽略振荡的插值函数 (虚线，式 (111))。蓝色区域表示小半径极限与大半径极限之间的过渡区: 在该区域内，两种极限解 (虚线和实线) 均不成立 [42]

and is displayed in right panel of Fig. 12, together with the other two approximate expressions of $G_\infty(r)$ derived above [42]. The interpolating function is an exponential, which is expected from the self-consistent implementation of the RG improvement (cf. section "Self-Consistency and Iterative RG Improvement") and from the tentative relation with the exponential nonlocal form factors potentially responsible for the damped oscillations of the lapse function. The exponential nature of the lapse function is also highly desirable since polynomial asymptotic scalings are not always compatible with the principle of least action [146]. We shall come back to this topic in section "Quantum Gravity Constraints from the Principle of Least Action."

且该内容展示在图 12 的右面板中，同时展示的还有前文推导得到的 $G_\infty(r)$ 的另外两种近似表达式 [42]。插值函数为指数形式，这是通过自洽实现 RG 改进得到的预期结果 (参见“自洽性与迭代 RG 改进”一节)，也符合它与指数型非定域形状因子的 tentative 关系——这类形状因子可能是 lapse 函数阻尼振荡的来源。lapse 函数的指数性质也十分理想，因为多项式渐近标度并不总能与最小作用量原理兼容 [146]。我们将在“最小作用量原理给出的量子引力约束”一节中回到这一话题。

Toward Black Holes from First Principles

从第一性原理出发研究黑洞

In this section we will summarize four independent investigations constraining aspects of quantum black holes, within and beyond asymptotic safety, hinging on first-principle calculations or considerations involving the FRG, the path integral, or the effective action. We shall present the corresponding findings in four subsections, following a chronological order.

本节我们将概述四项独立研究，这些研究基于第一性原理计算，或是利用 FRG、路径积分、有效作用量开展分析，在渐近安全框架内外限定了量子黑洞的部分性质。我们会按时间顺序在四个小节中依次介绍对应研究结果。

State Counting and Entanglement Entropy in Asymptotic Safety

渐近安全中的态计数与纠缠熵

In this subsection we review key findings on the topic of black hole entropy in asymptotically safe gravity. See also [55, 58, 59] for complementary works on the wider subject of black hole thermodynamics in asymptotic safety.

在本小节中，我们综述渐近安全引力框架下黑洞熵相关主题的核心研究结果。关于渐近安全中黑洞热力学这一更广泛议题的补充研究可参见 [55, 58, 59] 。

Microstate Counting [74, 147, 148]

微态计数 [74, 147, 148]

The scope of [74, 147, 148] was to provide an explanation for the area-scaling of the Bekenstein-Hawking entropy in terms of microscopic gravitational degrees of freedom. Becker and Reuter did so for Schwarzschild black holes in [147, 148] by proposing a state-counting formula based on the EAA Γ_k . Their calculation was subsequently applied to other types of black holes in [74].

[74, 147, 148] 的研究目标是从微观引力自由度的角度，解释贝肯斯坦-霍金熵的面积标度律。Becker 和 Reuter 在文献 [147, 148] 中针对史瓦西黑洞完成了这一工作，他们提出了一个基于有效平均作用量 EAA Γ_k 的态计数公式。该计算随后在文献 [74] 中被推广应用到其他类型的黑洞。

The derivation is based on a (Euclidean) path-integral, on-shell representation of the EAA

该推导基于 EAA 的 (欧氏) 路径积分、在壳表示

$$\mathbb{Z}_k \equiv e^{-\Gamma_k\left[0,\bar{g}_k^{\rm sf}\right]} = \int \mathcal{D}\widehat{\Phi} e^{-\tilde{S}\left[\widehat{\Phi},\bar{g}_k^{\rm sf}\right]} e^{-\Delta_k S\left[\widehat{\Phi}\right]} \tag{112}$$

where the integration is restricted to fluctuations $\widehat{\Phi}$ whose average is zero, $\langle\widehat{\Phi}\rangle = 0$. Moreover, $\tilde{S}$ stands for the sum of the bare, gauge fixing, and ghost actions; $\Delta_k S\left[\widehat{\Phi}\right] \sim \int d^d x \sqrt{-\bar{g}}\widehat{\Phi} R_k \widehat{\Phi}$ is the regulator which also appears in Eq. (1). Moreover, $\bar{g}_k^{\rm sf}$ is a so-called self-consistent background, i.e., a solution to the scale-dependent field equations defined by Γ_k

其中积分仅对平均值为零的涨落 $\widehat{\Phi}$ 进行，$\langle\widehat{\Phi}\rangle = 0$ 。此外，$\tilde{S}$ 代表裸作用量、规范固定作用量与鬼作用量之和；$\Delta_k S\left[\widehat{\Phi}\right] \sim \int d^d x \sqrt{-\bar{g}}\widehat{\Phi} R_k \widehat{\Phi}$ 是正规化因子，也出现在式 (1) 中。同时，$\bar{g}_k^{\rm sf}$ 是所谓自洽背景，即满足由 Γ_k 定义的依赖能标的场方程的解

$$\left.\frac{\delta \Gamma_k\left[h;\bar{g}\right]}{\delta h_{\mu\nu}}\right|_{h=0,\bar{g}=\bar{g}_k^{\rm sc}} = 0, \tag{113}$$

that is also used as expansion point in the background field method, $g = \bar{g}_k^{\rm sc} + h$.

它也在背景场方法中被用作展开点，$g = \bar{g}_k^{\mathrm{sc}} + h$。

The analysis can then be made concrete by specifying the bare action and by picking one of its specific solutions as a background field. Focusing on the case of the Einstein-Hilbert truncation with vanishing cosmological constant, solutions are Ricci flat spacetimes, and (Euclidean) Schwarzschild black holes constitute a prime example. Accordingly, [74, 147, 148] fixed the self-consistent background to be a Euclidean Schwarzschild solution, with $r \in [r_s, \infty)$ and $t \in \left[0, \beta = 4\pi r_s \equiv T^{-1}\right]$.

我们可以通过指定裸作用量、选取该作用量的一个特定解作为背景场，使分析具体化。聚焦于宇宙学常数为零的爱因斯坦-希尔伯特截断，其解是里奇平坦时空，而(欧氏)史瓦西黑洞是典型例子。据此，文献 [74, 147, 148] 将自洽背景固定为欧氏史瓦西解，其中 $r \in [r_s, \infty)$ 和 $t \in \left[0, \beta = 4\pi r_s \equiv T^{-1}\right]$
。

As the Euclidean Schwarzschild solution is Ricci flat, the bulk action vanishes on-shell, and the only contribution to the partition function comes from the boundary terms, so that the entropy of modes with $p^2 > k^2$ is given by

由于欧氏史瓦西解是里奇平坦的，体作用量在壳上为零，配分函数的唯一贡献来自边界项，因此动量模为 $p^2 > k^2$ 的熵由下式给出

$$\mathcal{S} = -\ln \mathbb{Z}_k = -\frac{1}{8\pi G_k^{\partial}} \int_{\partial\mathcal{M}} \mathrm{d}^3 x \sqrt{\bar{H}} \left(\bar{K} - \bar{K}_0\right) + \dots = \frac{\beta r_s}{4 G_k^{\partial}} + \dots \tag{114}$$

where $\bar{K}$ is the extrinsic curvature on the induced spatial background $\bar{H}$ and G_k^{∂} is the boundary Newton coupling - defined as the Newton coupling appearing in front of the boundary terms. In the limit where all fluctuating modes are integrated out, $k \to 0$, and provided that one can find a suitable trajectory with $G_k^{\partial} \to G_0$, the above expression yields the standard form for Bekenstein-Hawking area law.

其中 $\bar{K}$ 是诱导空间背景 $\bar{H}$ 上的外曲率，G_k^{∂} 是边界牛顿耦合——即出现在边界项前的牛顿耦合。在积分掉所有涨落模的极限 $k \to 0$ 下，只要能找到一条满足 $G_k^{\partial} \to G_0$ 的合适轨迹，上述表达式就可以给出贝肯斯坦-霍金面积定律的标准形式。

Finiteness of Entanglement Entropy [149]

纠缠熵的有限性 [149]

In [149] the authors focused on another aspect of entanglement entropies, namely, their infamous UV quadratic divergences.

在文献 [149] 中，作者聚焦于纠缠熵的另一个方面，也就是其臭名昭著的紫外二次发散。

Let us consider a quantum system with Hilbert space given by the direct product of two Hilbert spaces, $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. If the system is in the pure state $|\psi\rangle$, its total density matrix is $\rho = |\psi\rangle\langle\psi|$, while the reduced one for the subsystem A is obtained by tracing over $\mathcal{H}_B$. The entanglement entropy between the two subsystems then equals the von Neumann entropy:

我们考虑一个量子系统，其希尔伯特空间由两个希尔伯特空间的直积给出，$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$。若系统处于纯态 $|\psi\rangle$，其总密度矩阵为 $\rho = |\psi\rangle\langle\psi|$，子系统 A 的约化密度矩阵可通过对 $\mathcal{H}_B$ 求迹得到。两个子系统间的纠缠熵即等于冯·诺依曼熵:

$$\mathcal{S} = -\operatorname{Tr}\left[\rho_A \log \rho_A\right] \equiv -\lim_{n\to 1} \frac{\partial}{\partial n} \operatorname{Tr}\left[\rho_A^n\right]. \tag{115}$$

Concretely, its evaluation makes use of the replica trick, which boils down to a relation between the entanglement entropy and certain partition functions. To illustrate how this works, let us consider a free quantum field on a Minkowski spacetime, and let us introduce a spatial surface $\sum$ at $x_\mu = 0$, such that at $t = 0$ the field's degrees of freedom are located either at $x < 0$ (subsystem A) or at $x > 0$ (subsystem B). Then, the entanglement entropy between the two is given by

具体而言，它的计算利用了 replica trick(复本技巧)，该方法可归结为纠缠熵与特定配分函数之间的关系。为说明这一过程，我们考虑闵可夫斯基时空中的自由量子场，并在 $x_\mu = 0$ 处引入空间曲面 $\sum$，使得在 $t = 0$ 处，场的自由度要么位于 $x < 0$ (子系统 A)，要么位于 $x > 0$ (子系统 B)。那么，二者之间的纠缠熵可表示为

$$\mathcal{S} = \left[1 + 2\pi \frac{d}{d\delta}\right] \log Z_\delta \Big|_{\delta=0} \propto A_{\sum}[\bar{g}]\, \Lambda_{UV}^2, \tag{116}$$

where Z_δ is the partition function of the quantum field on a conical spacetime with deficit angle $\delta, A_{\sum}[\bar{g}]$ is the proper area of the surface $\sum$ with respect to the background metric $\bar{g}$, and Λ_{UV} is a UV momentum cutoff which is to be removed. It is thus clear that the entanglement entropy diverges quadratically as $\Lambda_{UV} \to \infty$. The question tackled in [149] is whether these divergences are removed by switching quantum gravity fluctuations on, as in this case the rigid background would be replaced by a dynamical, curved, fluctuating spacetime.

其中 Z_δ 是量子场在带亏缺角的锥时空上的配分函数，$\delta, A_{\sum}[\bar{g}]$ 是曲面 $\sum$ 相对于背景度规 $\bar{g}$ 的固有面积，Λ_{UV} 是将要被移除的紫外动量截断。因此很明显，当 $\Lambda_{UV} \to \infty$ 时纠缠熵会二次发散。文献 [149] 处理的问题是: 开启量子引力涨落是否能消除这些发散，因为在这种情况下，刚性背景会被动态、弯曲、涨落的时空所取代。

According to [149], the key to understand quantum gravity-induced modifications to Eq. (116) is the observation that in a quantum-gravitational setting, $\bar{g}$ is replaced by a self-consistent, scale-dependent background, cf. Eq. (113). Here in particular the cosmological constant plays a crucial role. Focusing on the case of the Einstein-Hilbert truncation, Eq. (113) yields

根据文献 [149] 的观点，理解量子引力对式 (116) 修正的关键在于: 在量子引力框架中，$\bar{g}$ 会被自洽的、依赖标度的背景取代，参见式 (113)。此处宇宙常数尤其起到了关键作用。聚焦于爱因斯坦-希尔伯特截断的情况，式 (113) 给出

$$R^\mu_\upsilon\left(\bar{g}_k^{\rm sc}\right) - \frac{1}{2}\delta^\mu_\upsilon R\left(\bar{g}_k^{\rm sc}\right) + \Lambda_k \delta^\mu_\upsilon = 0. \tag{117}$$

Solutions to this equation are such that $\Lambda_k \bar{g}_k^{\rm sc} = $ const and thus, in particular,

该方程的解满足 $\Lambda_k \bar{g}_k^{\mathrm{sc}} =$ 为常数，因此尤其可得

$$\left(\bar{g}_k^{\mathrm{sc}}\right)_{\alpha\beta} = \frac{\Lambda_\mu}{\Lambda_k}\left(\bar{g}_\mu^{\mathrm{sc}}\right)_{\alpha\beta} = \frac{\mu^2\lambda_\mu}{k^2\lambda_k}\left(\bar{g}_\mu^{\mathrm{sc}}\right)_{\alpha\beta}. \tag{118}$$

where μ is an arbitrary scale. The quadratic dependence on the momenta k and μ can at this point be exploited to redefine the metric in terms of its dimensionless counterpart that we shall call $\tilde{\bar{g}}$:

其中 μ 是任意标度。此时可以利用动量 k 和 μ 的二次依赖关系，用我们称之为 $\tilde{\bar{g}}$ 的无量纲度规重新定义原度规:

$$\left(\tilde{\bar{g}}_k^{\mathrm{sc}}\right)_{\alpha\beta} = \frac{\lambda_\mu}{\lambda_k}\left(\tilde{\bar{g}}_\mu^{\mathrm{sc}}\right)_{\alpha\beta}. \tag{119}$$

Thanks to the asymptotic safety condition, for which λ_k approaches the constant value λ_* asymptotically, the above equation is well defined in the limit $k \to \infty$. Together with the observation that the proper area function is linear, $A_\sum[c\bar{g}] = cA_\sum[\bar{g}]$, one finds that in asymptotic safety the limit [149]

得益于渐近安全条件——λ_k 渐近趋近于常数 λ_*，上述方程在 $k \to \infty$ 极限下是良定义的。结合固有面积函数是线性的这一结论，即 $A_\sum[c\bar{g}] = cA_\sum[\bar{g}]$，可以发现在渐近安全中，文献 [149] 得到极限

$$\mathcal{S} \propto \lim_{\Lambda_{UV}\to\infty} A_\sum\left[\bar{g}^{sc}_{\Lambda_{UV}}\right]\Lambda_{UV}^2 = A_\sum\left[\tilde{\bar{g}}^{sc}_*\right], \tag{120}$$

i.e., the entanglement entropy is finite. Moreover, by means of Eq. (119), the above expression can be re-written as

即纠缠熵是有限的。此外，通过式 (119)，上述表达式可以改写为

$$\mathcal{S} \propto \frac{\Lambda_\mu}{\lambda_*} A_\sum\left[\tilde{\bar{g}}^{sc}_\mu\right], \tag{121}$$

Keeping in mind that the total entanglement entropy ought to be independent of the normalization scale μ , one may fix it arbitrarily. Specifically, setting $\mu = m_{Pl} = G_0^{-1/2}$, one finds an expression that is close to the Bekenstein-Hawking formula and resembles the result (114) discussed above and first found in [147].

记住总纠缠熵应当与正规化标度 μ 无关，因此我们可以任意固定该标度。具体来说，当取 $\mu = m_{Pl} = G_0^{-1/2}$ 时，我们得到一个接近贝肯斯坦-霍金公式的表达式，与上文讨论的、最早由文献 [147] 得到的结果 (114) 一致。

Toward a Dressed Newtonian Potential Within Asymptotic Safety

渐近安全框架内的修饰牛顿势研究

The metric of Schwarzschild or structurally similar modified black holes can be written in terms of a lapse function

史瓦西度量或结构类似的修正黑洞度量可以用时移函数表示

$$f(r) = 1 - 2\Phi(r), \tag{122}$$

where $\Phi(r)$ is sometimes dubbed "Newtonian potential." It is important to remark that this a slight abuse of notation, since the identification of $\Phi(r)$ with the classical Newtonian potential - defined as the potential exerted by a massive point source on a test particle (hence, non-interacting and non-massive, so as not to impact the spacetime) - only holds asymptotically, in the weak field regime.

其中 $\Phi(r)$ 有时被称为“牛顿势”。需要指出，这是对记号的一种轻微误用: 只有在弱场渐近区域，$\Phi(r)$ 才等同于经典牛顿势——经典牛顿势被定义为大质量点源对试验粒子产生的势，试验粒子无相互作用且质量可忽略，因此不会影响时空。

In a quantum gravity context, this pseudo-Newtonian potential ought to be determined by solving the quantum field equations (4). In turn, this would require the computation of a sufficiently accurate approximation to the effective action.

在量子引力语境下，这一伪牛顿势需要通过求解量子场方程 (4) 得到，而这要求我们计算足够精确的有效作用量近似。

While such an expression has not been computed yet, one may get an intuition on the form of the Newtonian potential by calculating the graviton-mediated $2 \to 2$ scattering amplitude of two scalar fields minimally coupled to gravity, in the static limit. At one loop, this reproduces the well-known result by Donoghue on the leading-order corrections to the Newtonian potential [61]. First steps toward the extension of Donoghue's calculation beyond one-loop [6] and within asymptotic safety have been taken in [34]. The scope of this subsection is to summarize the work of [34] and its conclusions.

虽然目前还没有得到这样的表达式，但我们可以通过计算静态极限下，最小耦合于引力的两个标量场之间由引力子传递的 $2 \to 2$ 散射振幅，来推测牛顿势的形式。一圈计算可以重现多诺霍对牛顿势领头阶修正的著名结果 [61]。在渐近安全框架内，将多诺霍的计算拓展到多圈 [6] 的第一步工作已经在文献 [34] 中完成。本小节的目的是总结文献 [34] 的工作及其结论。

Prior to starting, it is important to remark that the identification of the Newtonian potential with the one stemming from a graviton-mediated $2 \to 2$ scattering between matter or gauge fields, which we shall denote by $V(r)$, might not hold in general. On the formal side, indeed, the first is the potential exerted by a massive static source on a test particle, while the second is the potential between two massive and interacting fields. On the practical side, there exists already a clear counterexample where this identification does not work [151]: when introducing a cosmological constant, the Newtonian potential is a simple asymptotically de Sitter spacetime, whereas the potential from a $2 \to 2$ scattering amplitude is characterized by a de Sitter horizon and vanishes beyond it. Given the argument as well as the counterexample in [151], it is unlikely that the two potentials will coincide in a full-fledged quantum gravity computation. Yet, determining $V(r)$ is important on several levels, including the comparison with the EFT results [61], and could provide some hints on singularity resolution in asymptotic safety [34].

在开始之前，必须指出，由物质场或规范场之间引力子传递 $2 \to 2$ 散射得到的牛顿势 (我们将其记为 $V(r)$) 并不一定符合原本的定义。从形式上看，前者是大质量静态源对试验粒子产生的势，而后者是两个大质量相互作用场之间的势；实际中也已经存在明确的反例说明这一等价不成立 [151]: 引入宇宙学常数时，牛顿势对应渐近德西特时空，而 $2 \to 2$ 散射振幅得到的势存在德西特视界，在视界外该势为零。结合文献 [151] 中的论证与反例，在完整的量子引力计算中，这两个势不太可能等同。但即便如此，确定 $V(r)$ 在多个层面上都有重要意义，包括和有效场论结果 [61] 对比，还能为渐近安全框架下的奇点消解问题提供思路 [34]。

The work in [34] considers the effective gravitational scattering of two scalar fields minimally coupled to gravity. The calculation is performed in the static approximation, where both scalar fields are infinitely massive, $m_i \to \infty$. In this limit the potential between the two scalar masses reads [34,61]

文献 [34] 中的工作研究了最小耦合于引力的两个标量场之间的有效引力散射。计算在静态近似下完成，即两个标量场质量都趋近于无穷大，$m_i \to \infty$ 。在该极限下，两个标量质量之间的势可写为 [34,61]

$$V(r) = -\frac{1}{2m_1}\frac{1}{2m_2}\int \frac{d^3\mathbf{q}}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{r}} \mathcal{M}\left(\mathbf{q}^2\right), \tag{123}$$

where $\mathcal{M}$ is the scattering amplitude of two scalars into two scalars, mediated by one graviton line

其中 $\mathcal{M}$ 是单个引力子传递的两个标量到两个标量的散射振幅

$$\mathcal{M}\left(\mathbf{q}^2\right) = 16\pi G_0 m_1^2 m_2^2 \mathcal{G}\left(\mathbf{q}^2\right), \tag{124}$$

with $\mathcal{G}\left(\mathbf{q}^2\right)$ being the scalar part of the (dressed) graviton propagator in the nonrelativistic limit $q \to (0, \mathbf{q})$. For small momenta, one may neglect higher-derivative operators in the effective action, and the scaling of the propagator reduces to the Einstein-Hilbert one, $\mathcal{G} \propto 1/q^2$. Correspondingly, on large-distance scales the leading-order scaling of the exact potential $V_{qu}(r)$ is well approximated by its classical counterpart, $V_{cl}(r) \propto 1/r$; in this subsection we explain the steps taken in [34] to derive a quantum version of V_{cl} within asymptotic safety. The technical difficulties involved in the computation of V_{qu} require however a number of approximations, on top of the static limit, that we detail in the following.

其中 $\mathcal{G}\left(\mathbf{q}^2\right)$ 是非相对论极限下 (修饰) 引力子传播子的标量部分 $q \to (0, \mathbf{q})$ 。对于小动量，可以忽略有效作用量中的高阶导数算符，传播子的标度退化为爱因斯坦-希尔伯特形式，$\mathcal{G} \propto 1/q^2$ 。相应地，在大尺度上，精确势 $V_{qu}(r)$ 的领头阶标度可以很好地用经典形式 $V_{cl}(r) \propto 1/r$ 近似；本小节我们将介绍文献 [34] 中，在渐近安全框架内推导 V_{cl} 量子形式的步骤。但由于计算 V_{qu} 存在诸多技术困难，除了静态极限外还需要引入一系列近似，我们将在下文逐一说明。

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[6] It is worth mentioning that the one-loop results in [61] have not been reproduced yet via full-fledged FRG computations. See however [150] for first steps in this direction.

[6] 值得一提的是，文献 [61] 中的一圈结果目前还未能通过完整的泛函重整化群计算重现，不过文献 [150] 已经给出了该方向的初步工作。

V_{qu} is determined by the full momentum dependence of the dressed propagator. In turn, this is encoded in the form factors $g_R(\Box)$ and $g_C(\Box)$ in the effective action (3). Focusing on the contribution from the transverse-traceless (TT) spin-2 mode, the scalar part of the dressed TT propagator reads

> V_{qu} 由修饰传播子的完整动量依赖关系决定，而这又被编码在有效作用量 (3) 中的形状因子 $g_R(\Box)$ 和 $g_C(\Box)$ 里。聚焦于无迹横波 (TT) 自旋 2 模式的贡献，修饰 TT 传播子的标量部分可写为

$$\mathcal{G}^{TT}\left(q^2\right) = q^{-2}\left(1 + 2q^2 g_C\left(q^2\right)\right)^{-1}. \tag{125}$$

In particular, the function $g_C\left(q^2\right)$ is to be computed from first principles, e.g., either by solving the path integral or by determining the IR limit, $k \to 0$, of its flowing counterpart $g_{C,k}\left(q^2\right)$. The latter strategy involves inserting the ansatz (2) in the FRG equations (1), deriving the beta functions for all couplings in it, finding a suitable solution (i.e., an RG trajectory) departing from a UV fixed point and reaching an IR limit compatible with general relativity, and determining the limit $\lim\limits_{k\to 0} g_{C,k}\left(q^2\right)$. Every single step in this procedure is technically highly involved.

> 特别地，函数 $g_C\left(q^2\right)$ 需从第一性原理计算得到，例如通过求解路径积分，或是确定其流动对应量 $g_{C,k}\left(q^2\right)$ 的红外极限 $k \to 0$ 。后一种策略需要将假设 (2) 代入 FRG 方程 (1)，推导出其中所有耦合的 β 函数，找到一条合适的解 (即 RG 轨迹)——从紫外不动点出发，最终到达与广义相对论相容的红外极限，再确定极限 $\lim\limits_{k\to 0} g_{C,k}\left(q^2\right)$ 。该流程的每一步在技术上都极具复杂性。

In order to make a first evaluation of $V_{qu}(r)$ in asymptotic safety possible,[34] considered terms in the EAA (2) up to quadratic order in a curvature expansion, neglected the form factor $g_R(\Box)$, exploited an expansion around flat space, and on the right-hand side of the flow equation (1) considered the fluctuations of the conformal mode only. Under these approximations, the projection of the exact FRG equation (1) on the subspace of theories spanned by $\left\{\Lambda_k, G_k, g_{C,k}\left(q^2\right)\right\}$ yields the beta functions for the corresponding dimensionless couplings $\{\lambda_k, g_k\}$ and dimensionless form factor $w_k\left(q^2\right) = k^{-2} g_{C,k}\left(q^2\right)$ [34]. Their non-trivial fixed points are determined by the condition

> 为了能够在渐近安全框架下首次计算 $V_{qu}(r)$ ，文献 [34] 将有效平均作用量 (2) 中的项考虑到曲率展开的二次阶，忽略了形状因子 $g_R(\Box)$ ，利用了平坦空间附近的展开，并且在流方程 (1) 的右侧仅考虑共形模式的涨落。在这些近似下，将精确 FRG 方程 (1) 投影到 $\left\{\Lambda_k, G_k, g_{C,k}\left(q^2\right)\right\}$ 张成的理论子空间，得到了对应无量纲耦合 $\{\lambda_k, g_k\}$ 和无量纲形状因子 $w_k\left(q^2\right) = k^{-2} g_{C,k}\left(q^2\right)$ 的 β 函数 [34]。它们的非平凡不动点由以下条件确定

$$k\partial_k g_k = k\partial_k \lambda_k = k\partial_k w_k\left(q^2\right) = 0. \tag{126}$$

that is a system of coupled differential and integro-differential equations which can be solved numerically. The Einstein-Hilbert sector spanned by the dimensionless couplings $\{\lambda_k, g_k\}$ turns out to be independent of the last equation and can thereby be solved independently. It admits a non-trivial fixed point at $\{\lambda_* = 0.285, g_* = 0.374\}$ which can act as a UV attractor. Replacing these values in the integrodifferential equation for $w_k\left(q^2\right)$, one can finally determine the fixed-function $w_*\left(q^2\right)$ numerically. Figure 13 shows both the numerical solution (solid, purple line) and its analytic approximation (dashed, pink line)

即这是一个耦合微分方程与积分微分方程构成的方程组，可以数值求解。结果表明，无量纲耦合 $\{\lambda_k, g_k\}$ 张成的爱因斯坦-希尔伯特扇区与最后一个方程无关，因此可以单独求解。它在 $\{\lambda_* = 0.285, g_* = 0.374\}$ 处存在一个非平凡不动点，可作为紫外吸引子。将这些数值代入 $w_k(q^2)$ 的积分微分方程，最终就可以数值确定不动函数 $w_*(q^2)$。图 13 同时给出了数值解 (紫色实线) 及其解析近似 (粉色虚线)

$$w_*^{\mathrm{fit}}(q^2) \approx w_\infty + \frac{\kappa\rho}{\rho + \kappa q^2}, \tag{127}$$

where $\rho \approx 0.0149, \kappa \approx 0.00817$, and w_∞ are free parameters. Its structure indicates the presence of nonlocalities in the bare action - a feature that may allow to explain the area law of black holes within a quantum field theoretic setup [152].

其中 $\rho \approx 0.0149, \kappa \approx 0.00817$ 和 w_∞ 为自由参数。其结构表明裸作用量中存在非定域性，这一特性或许能够在量子场论框架下解释黑洞的面积定律 [152]。

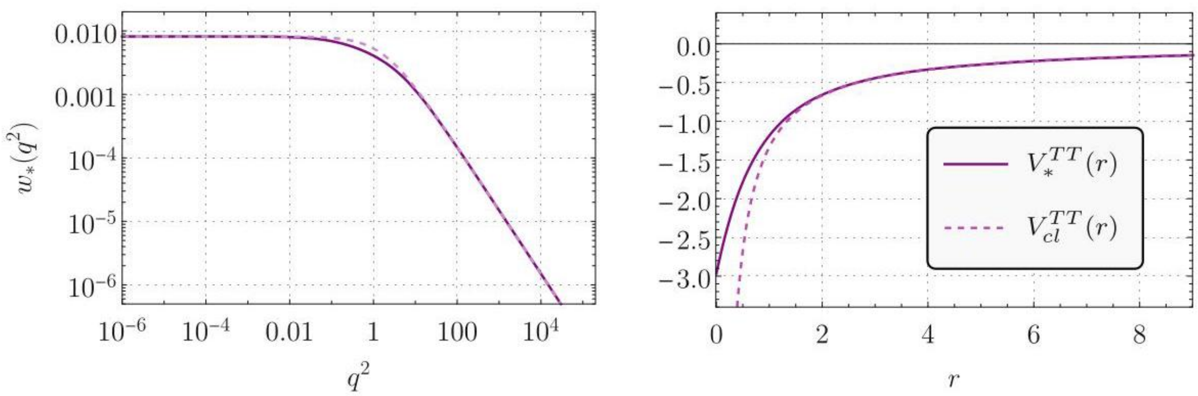

Fig. 13 Dimensionless UV form factor $w_*(q^2)$ (left panel) and the corresponding dimensionful TT potential $V_*^{TT}(r)$ in coordinate space (right panel). Both are depicted with a solid purple line. The first is obtained numerically. The analytical approximation of $w_*(q^2)$ in Eq. (127) is also shown for w_∞ in the left panel (dashed, violet line). The second one is obtained by using Eq. (123) and the TT propagator (127) with $w_\infty = 0.1$. The classical TT potential $V_{cl}^{TT}(r)$ is shown as well for comparison (dashed, magenta line). For $w_\infty > 0$, the fixed-point potential $V_*^{TT}(r)$ is finite at $r = 0$, while the classical one diverges in the same limit, resembling the behavior of $\Phi(r)$ for classical black holes. Both figures have been obtained by using the numerical data in [34]

图 13 无量纲紫外形状因子 $w_*(q^2)$ (左图) 与坐标空间中对应的量纲 TT 势 $V_*^{TT}(r)$ (右图)。二者均由紫色实线绘出。其中 $w_*(q^2)$ 由数值计算得到。左图中还以紫色虚线给出了式 (127) 中 $w_*(q^2)$ 对 w_∞ 的解析近似。$V_*^{TT}(r)$ 则通过式 (123) 和带有 $w_\infty = 0.1$ 的 TT 传播子 (127) 计算得到。经典 TT 势 $V_{cl}^{TT}(r)$ 也以品红色虚线标出，用于对比。在 $w_\infty > 0$ 下，固定点势 $V_*^{TT}(r)$ 在 $r = 0$ 处有限，而经典势在同一极限下发散，这与经典黑洞的 $\Phi(r)$ 行为类似。两幅图均使用文献 [34] 中的数值数据绘制而成

Assuming that neither the fluctuations beyond the conformal part nor the flow from the UV fixed point to the physical limit $k \to 0$ substantially modify the form of the fixed-point propagator, one may approximate $\mathcal{G}\left(q^2\right)$ with the scalar part of the TT graviton propagator at the UV fixed point, and correspondingly

假设共形部分之外的涨落，以及从紫外固定点到物理极限 $k \to 0$ 的流都不会大幅改变固定点传播子的形式，我们就可以用紫外固定点处 TT 引力子传播子的标量部分来近似 $\mathcal{G}\left(q^2\right)$，相应地

$$V_{qu}(r) \approx V_*^{TT}(r). \tag{128}$$

This potential can be computed numerically by combining Eq. (123) with the TT propagator (127) (cf. Fig. 13) and is to be confronted with the corresponding TT part of the potential in general relativity:

该势可以通过结合式 (123) 与 TT 传播子 (127) 数值计算得到 (参见图 13)，需要与广义相对论中该势对应的 TT 部分进行对比:

$$V_{cl}(r) = -\frac{4}{3}\frac{1}{r}. \tag{129}$$

At variance of the case of general relativity, and under all approximations detailed above, quantum gravity effects in asymptotic safety make the scattering potential of two scalars finite at $r = 0$: if a similar mechanism applies to black hole and cosmological solutions, it would imply that spacetime singularities are weakened in asymptotic safety.

与广义相对论的情况不同，在上述所有近似下，渐近安全中的量子引力效应使得两个标量的散射势在 $r = 0$ 处有限: 如果类似的机制适用于黑洞和宇宙学解，那么这意味着时空奇点在渐近安全框架中会被消解。

Constraints from the Gravitational Path Integral: Bare Actions and Dynamical Singularity-Resolution Mechanism

引力路径积分的约束: 裸作用量与动力学奇异性 Resolution 机制

In this subsection we summarize the findings in [153], where the authors - inspired by the arguments of [154] in a cosmological context - discuss a "dynamical" black hole singularity-resolution mechanism based on the suppression of singular spacetime configurations in the gravitational path integral and its role in constraining the form of the bare action S_{bare} .

在本小节中，我们总结文献 [153] 的研究结果: 该文作者受宇宙学背景下文献 [154] 论点的启发，讨论了基于引力路径积分中奇异性时空构型压制的“动力学”黑洞奇异性 Resolution 机制，及其在约束裸作用量形式中的作用 S_{bare} 。

The solutions $g_{\mu\nu}^{\text{sol}}$ to the quantum field equations (4) are equivalently defined as expectation values according to

量子场方程 (4) 的解 $g_{\mu\nu}^{\rm sol}$ 等价地定义为下述期望值

$$g_{\mu\nu}^{sol} \equiv \int \mathcal{D}g g_{\mu\nu} e^{iS_{\rm bare}[g]}, \tag{130}$$

with different solutions corresponding to different initial conditions in the gravitational path integral. Every spacetime configuration $g_{\mu\nu}$ comes with a "Lorentzian weight" $e^{iS_{\rm bare}[g]}$. In turn, the latter corresponds to a statistical weight $e^{-S_{\rm bare}[g]}$ in the Euclidean version of the gravitational path integral:

不同解对应引力路径积分中不同的初始条件。每个时空构型 $g_{\mu\nu}$ 都带有一个“洛伦兹权重” $e^{iS_{\rm bare}[g]}$ ，而该权重对应欧几里得型引力路径积分中的统计权重 $e^{-S_{\rm bare}[g]}$:

$$\mathcal{Z}_E \equiv \int \mathcal{D}g^E e^{-S_{\rm bare}\left[g^E\right]}. \tag{131}$$

A given solution $g_{\mu\nu}^{\rm sol}$ can be seen as a superposition of spacetime configurations. In particular, the configurations contributing the most are those making the bare action small. By contrast, spacetimes for which the bare action is divergent are suppressed in the Euclidean gravitational path integral; at the Lorentzian level, this suppression translates in a fast-oscillating weighting factor producing destructive interference. This simple consideration can be exploited to investigate the singularity resolution mechanism in quantum gravity [153,154]: in order to produce regular solutions $g_{\mu\nu}^{\rm sol}$, one needs all singular configurations in the path integral to be suppressed. As a consequence, a theory of quantum gravity can allow for singularity resolution if its bare action diverges when evaluated on singular spacetimes. Based on these arguments, the authors of [153] focused on black hole configurations and investigated what minimal conditions on the bare action $S_{\rm bare}$ are such that this dynamical black hole singularity resolution is possible.

任意给定解 $g_{\mu\nu}^{\rm sol}$ 可被视为多个时空构型的叠加。具体而言，贡献最大的构型是那些让裸作用量取小值的构型。反之，裸作用量发散的时空会在欧几里得引力路径积分中被压制；在洛伦兹视角下，这种压制对应快速振荡的权重因子，产生相消干涉。这一简单思路可用于研究量子引力中的奇异性 Resolution 机制 [153, 154]: 要得到正则解 $g_{\mu\nu}^{\rm sol}$ ，需要路径积分中所有奇异性构型都被压制。因此，若量子引力理论的裸作用量在奇异性时空上求值时发散，该理论就可以实现奇异性 Resolution。基于这些论点，文献 [153] 的作者聚焦黑洞构型，研究了要实现这种动力学黑洞奇异性 Resolution，裸作用量 $S_{\rm bare}$ 需要满足的最低条件。

Given a spacetime $g_{\mu\nu}^{\rm sing}$ characterized by a curvature singularity, the divergence of the corresponding bare action $S_{\rm bare}\left[g^{\rm sing}\right]$ depends on the curvature invariants appearing in it, as some of them could remain finite despite the singularities of $g^{\rm sing}$. For instance, the Ricci scalar R vanishes for Schwarzschild spacetimes, and therefore the Einstein-Hilbert action and its $f(R)$ -like extensions do not diverge on a large class of singular spacetimes. This simple consideration implies that higher-derivative terms beyond $f(R)$ -like models are required to allow for singularity resolution. In particular, since invariants built on the Ricci tensor $R_{\mu\nu}$ also vanish for Ricci-flat spacetimes, singularity resolution would require at least some terms built from the Riemann tensor, e.g., $R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$. A bare action based on Stelle gravity

对于存在曲率奇异性的给定时空 $g_{\mu\nu}^{\text{sing}}$ ，对应裸作用量 $S_{\text{bare}}\left[g^{\text{sing}}\right]$ 是否发散取决于其中出现的曲率不变量——即便 g^{sing} 存在奇异性，部分曲率不变量仍可能保持有限。例如，史瓦西时空的里奇标量 R 等于零，因此爱因斯坦-希尔伯特作用量及其类 $f(R)$ 拓展不会在一大类奇异性时空上发散。这一简单结论说明，要实现奇异性 Resolution，需要类 $f(R)$ 模型之外的更高导数项。具体而言，由于里奇张量 $R_{\mu\nu}$ 构造的不变量在里奇平坦时空中也等于零，奇异性 Resolution 至少需要一些由黎曼张量构造的项，例如 $R_{\mu\upsilon\sigma\rho}R^{\mu\upsilon\sigma\rho}$ 。基于斯泰勒引力的裸作用量

$$S_{\text{bare}}[g]=\int d^4x\sqrt{-g}\left(\frac{2\Lambda-R}{16\pi G}+aR^2+bR_{\mu\upsilon\sigma\rho}R^{\mu\upsilon\sigma\rho}\right),\tag{132}$$

is therefore the minimal extension beyond Einstein-Hilbert that can in principle allow for singularity resolution [153] (and, in a cosmological context, to allow for a suppression of inhomogeneous and anisotropic configurations, over those satisfying the cosmological principle [154]) grounded on the dynamical singularity-resolution mechanism described above.

因此是爱因斯坦-希尔伯特作用量之外的最小拓展，原则上可以支持基于上述动力学奇异性 Resolution 机制的奇异性 Resolution[153](而在宇宙学背景下，它可以压制非均匀、各向异性构型，偏好满足宇宙学原理的构型 [154])。

Quantum Gravity Constraints from the Principle of Least Action

来自最小作用量原理的量子引力约束

Lacking a direct derivation from quantum gravity, several spacetime models have been proposed that replace classical singular black holes while reproducing their Schwarzschild exteriors at large distances. These models include, for instance, regular black holes [49,88,134,155], black holes with integrable singularities [156], wormholes [157-159], and a variety of compact objects [160-165].

由于缺乏从量子引力出发的直接推导，目前已有多个时空模型被提出，这类模型替换了经典的奇异黑洞，同时能在大距离下还原其史瓦西外部解。这类模型包括正则黑洞 [49,88,88,134,155]、具有可积奇点的黑洞 [156]、虫洞 [157-159]，以及各类致密天体 [160-165]。

As argued in [146], a basic requirement for these ad hoc models to be physical, i.e., to have a fundamental explanation in terms a complete theory of quantum gravity and matter, is that the corresponding metric $g_{\mu\upsilon}$ ought to be a solution to effective field equations (4) stemming from an effective action Γ_0 . In particular, one could ask the following question: given a spacetime described by a metric $\bar{g}_{\mu\upsilon}$, is there an effective action Γ_0 such that

正如文献 [146] 所述，这些特设模型要具备物理合理性，也就是能在完整的量子引力与物质理论中得到基础性解释，一个基本要求是: 对应度规 $g_{\mu\upsilon}$ 必须是源自有效作用量 Γ_0 的有效场方程 (4) 的解。具体来说，我们可以提出这样一个问题: 给定一个由度规 $\bar{g}_{\mu\upsilon}$ 描述的时空，是否存在一个有效作用量 Γ_0 ，使得

$$\left.\frac{\delta\Gamma_0\,[g]}{\delta g_{\mu\nu}}\right|_{g=\bar{g}} = 0? \tag{133}$$

In general, this task is extremely involved. Yet, substantial progress can already be made by focusing on the asymptotic region $r \to \infty$ and by exploiting a relatively general parametrization of the corrections to the Schwarzschild scaling [146]. This strategy also comes with several advantages:

一般而言，这项工作极为复杂。但只需聚焦渐近区域 $r \to \infty$ ，并利用对史瓦西标度修正的相对通用参数化 [146]，就能取得可观进展。这种策略还有诸多优势:

- The constraints resulting from the asymptotic analysis ought to apply to any theory of quantum gravity, since asymptotically all of them have to recover an EFT/QFT description where the principle of least action plays a crucial role.

- 渐近分析得到的约束适用于任意量子引力理论，因为任何量子引力理论在渐近区域都必须还原为有效场论/量子场论描述，而最小作用量原理在这类描述中发挥着核心作用。

- It allows to constrain a number of different proposed models.

- 该方法可以对多种已提出的模型给出约束。

In an attempt to answering the aforementioned question, the authors of [146] derived strong constraints on the asymptotic scaling of modified black holes beyond general relativity by enforcing the validity of the principle of least action. The scope of this subsection is to summarize their work and the resulting constraints.

为解答上述问题，文献 [146] 的作者通过 enforcing 最小作用量原理的有效性，对广义相对论之外的修正黑洞的渐近标度给出了强约束。本小节的范围是总结他们的工作与得到的约束。

Following [146], we shall focus on the asymptotic scaling of static, spherically symmetric spacetimes

沿用文献 [146] 的思路，我们将聚焦静态、球对称时空的渐近标度

$$\bar{g}_{\mu\nu} = \mathrm{diag}\left(-f_{tt}\,(r)\,, \frac{1}{f_{rr}\,(r)}, r^2, r^2 \sin\theta\right), \tag{134}$$

and we shall further assume the metric coefficients to admit an asymptotic expansion of the form

并进一步假设度规系数可以写成如下形式的渐近展开

$$f_{tt}\,(r) \sim 1 - \frac{2G_N m}{r} + \frac{c_t}{r^{n_t}},\; f_{rr}\,(r) \sim 1 - \frac{2G_N m}{r} + \frac{c_r}{r^{n_r}}, \tag{135}$$

where $n_r, n_t > 1$, such that $c_i r^{-n_i}$ are sub-leading, asymptotic corrections to the pure Schwarzschild scaling. This condition applies to the most commonly studied alternatives to Schwarzschild black holes, e.g., [49, 88, 155, 166], as they are built on ratios of polynomials. Specifically, the asymptotic scaling of the Bardeen, Bonanno-Reuter, and Hayward metric coefficients matches the one in Eq. (135), with $n_r = n_t = 4$ and specific values of c_r and c_t . By contrast, the asymptotic expansion (135) does not apply to black holes whose lapse functions contain exponentials of the radial coordinate, e.g., as in [134], or similar non-algebraic functions.

The constraints derived in [146] thus only apply to the former class of models. Note in particular that since the analysis in [146] is grounded on asymptotic considerations, it applies to any modification to the Schwarzschild metric, including singular black holes, and all sorts of black hole mimickers, including horizonless objects.

其中 $n_r, n_t > 1$ ，满足 $c_i r^{-n_i}$ 是对纯史瓦西标度的次导渐近修正。该条件适用于最常被研究的史瓦西黑洞替代模型，例如 [49, 88, 155, 166]，因为这些模型都基于多项式比构造。具体来说，巴丁、博南诺-罗伊特和海沃德度规系数的渐近标度都符合式 (135) 的形式，对应 $n_r = n_t = 4$ 以及 c_r 和 c_t 的特定取值。相比之下，渐近展开式 (135) 不适用于时移函数包含径向坐标指数项的黑洞 (例如文献 [134] 中的模型)，或是其他类似非代数函数的黑洞。因此文献 [146] 得到的约束仅适用于前一类模型。需要特别注意，由于文献 [146] 的分析基于渐近考虑，它适用于对史瓦西度规的任意修正，包括奇异黑洞，以及所有类型的黑洞仿制品，包括无视界天体。

At this point, in order to establish whether the metric (134), for some values of c_i and n_i , can come from a principle of least action, one can take the following steps:

至此，要判断式 (134) 的度规在 c_i 和 n_i 取某些值时，是否可以从最小作用量原理导出，我们可以遵循以下步骤:

1. Parametrize the effective action Γ_0 , e.g., using a curvature expansion,

1. 对有效作用量 Γ_0 做参数化，例如使用曲率展开，

$$\Gamma_0 = \frac{1}{16\pi G_N} \int \mathrm{d}^4 x \sqrt{-g} \left[-R - \frac{1}{6} R f_R(\Box) R + R^{\mu}{}_{\nu} f_{Ric}(\Box) R^{\nu}{}_{\mu} + O\left(\mathcal{R}^3\right) \right], \tag{136}$$

and truncate to a certain order. Here $\Box = -g^{\mu\upsilon} D_\mu D_\upsilon$ is the d’Alembert operator built from a metric $g, \mathcal{R}$ stands for a generic curvature invariant, and $f_i(\Box)$ are form factors, similarly to those we introduced in Eq. (3), that encode the physical momentum dependence of the gravitational couplings [24, 25, 31-35]. Determining them from first principles ought to be a task of every approach to quantum gravity.

并截断到某一阶。此处 $\Box = -g^{\mu\upsilon} D_\mu D_\upsilon$ 是由度规构造的达朗贝尔算符，$g, \mathcal{R}$ 代表通用曲率不变量，$f_i(\Box)$ 是形状因子，和我们在式 (3) 中引入的类似，它们编码了引力耦合的物理动量依赖 [24, 25, 31-35]。从第一性原理出发确定这些形状因子，是所有量子引力研究方法都需要完成的任务。

2. Derive the corresponding field equations (4) and evaluate them on the ansatz (134) with metric coefficients (135). The resulting equations will be dependent on (c_i, n_i) as well as on the couplings g_i in Γ_0 . Structurally, each equation will take the form

2. 推导对应的场方程 (4)，并将其代入带有度规系数 (135) 的拟设 (134) 求值。得到的方程将依赖于 (c_i, n_i) ，以及 Γ_0 中的耦合 g_i 。从结构上看，每个方程都可以写成如下形式

$$\sum_{j > \bar{j}} a_j (g_i, c_i, n_i) r^{-j} = 0, \tag{137}$$

where $a_j(g_i, c_i, n_i)$ are functions of the couplings g_i and the parameters (c_i, n_i) , while $\bar{j} \geq 0$ is an integer whose specific value depends on the form of Γ_0 .

其中 $a_j\,(g_i, c_i, n_i)$ 是耦合常数 g_i 和参数 (c_i, n_i) 的函数，而 $\bar{j} \geq 0$ 是一个整数，其具体取值取决于 Γ_0 的形式。

3. Establish if there exists (g_i, c_i, n_i) such that the equations are identically fulfilled to leading order in the above asymptotic expansion, $a_{\bar{j}}\,(g_i, c_i, n_i) = 0$. Note that this constitutes only a necessary condition for $\bar{g}_{\mu\nu}$ to be a solution. Next-to-leading order terms ought to be considered in the ansatz (135) (by construction these terms will not contribute to the leading-order coefficient $a_{\bar{j}}$) and in Eq. (137). It turns out that the leading-order condition $a_{\bar{j}}\,(g_i, c_i, n_i) = 0$ will suffice to put strong constraints on either (c_i, n_i) or Γ_0 .

3. 确认是否存在 (g_i, c_i, n_i) 使得这些方程在上述渐近展开 $a_{\bar{j}}\,(g_i, c_i, n_i) = 0$ 的领头阶完全满足。注意这只是 $\bar{g}_{\mu\nu}$ 成为解的必要条件。次领头阶项需要在假设 (135)(根据构造，这些项不会对领头阶系数 $a_{\bar{j}}$ 产生贡献) 和式 (137) 中予以考虑。结果表明，仅领头阶条件 $a_{\bar{j}}\,(g_i, c_i, n_i) = 0$ 就足以对 (c_i, n_i) 或 Γ_0 给出强约束。

The first question to ask is whether the modified metric (134) considered in [146] can be a solution to field equations stemming from a local effective action Γ_0 . We shall thus first consider a local version of (136) by expanding the form factors $f_i\,(\Box)$ in a derivative expansion involving only positive power of the d' Alembertian operator. The resulting effective action thus reads

我们首先需要回答的问题是，文献 [146] 中考虑的修正度规 (134) 能否成为源自局部有效作用量 Γ_0 的场方程的解。因此，我们首先对 (136) 取局部形式，将形状因子 $f_i\,(\Box)$ 展开为仅包含达朗贝尔算符正幂次的导数展开。由此得到的有效作用量形式为

$$\Gamma_0^{loc} = \Gamma_{GR} + \Gamma_{\mathcal{R}^2} + \Gamma_{\mathcal{R}^3} + \dots, \tag{138}$$

with

其中

$$\Gamma_{GR} = -\frac{1}{16\pi G_N}\int \mathrm{d}^4x\sqrt{-g}R, \tag{139a}$$

$$\Gamma_{\mathcal{R}^2} = \frac{1}{16\pi G_N}\int \mathrm{d}^4x\sqrt{-g}\left[-\frac{a}{6}R^2 + bR^{\mu}{}_{\nu}R^{\nu}{}_{\mu}\right], \tag{139b}$$

$$\Gamma_{\mathcal{R}^3} = \frac{1}{16\pi G_N}\int \mathrm{d}^4x\sqrt{-g}\left[k_{RC^2}RC^{\mu\upsilon\rho\sigma}C_{\mu\upsilon\rho\sigma} + k_{C^3}C_{\mu\upsilon}{}^{\rho\sigma}C_{\rho\sigma}{}^{\tau\omega}C_{\tau\omega}{}^{\mu\upsilon}\right].$$

(139c)

The condition for the modified metric (134) to be a solution to the field equations associated with the local effective action (138) is

修正度规 (134) 成为局部有效作用量 (138) 对应场方程解的条件为

$$\left.\frac{\delta\Gamma_0^{loc}\,[g]}{\delta g_{\mu\upsilon}}\right|_{g=\bar{g}} = \left.\left(\frac{\delta\Gamma_{GR}\,[g]}{\delta g_{\mu\upsilon}} + \frac{\delta\Gamma_{\mathcal{R}^2}\,[g]}{\delta g_{\mu\upsilon}} + \frac{\delta\Gamma_{\mathcal{R}^3}\,[g]}{\delta g_{\mu\upsilon}} + \cdots + \frac{\delta\Gamma_{\mathcal{R}^N}\,[g]}{\delta g_{\mu\upsilon}}\right)\right|_{g=\bar{g}} = 0,$$

(140)

in an asymptotic expansion for large radii. In particular, the scaling of one of the higher-derivative terms with r ought to cancel general relativity contribution. Due to the symmetries of (134), the only two independent components of these field equations can be obtained by deriving every term in Γ_0 with respect to the metric components g_{rr} and g_{tt} . One can thus compute the individual asymptotic contributions of each term in Eq. (140). Setting $n = n_t = n_r$ for shortness, they read

在大半径的渐近展开中成立。特别地，其中一个高阶导数项与 r 相关的标度应当抵消广义相对论的贡献。由于 (134) 的对称性，我们可以通过对 Γ_0 中每一项关于度规分量 g_{rr} 和 g_{tt} 求导得到场方程仅有的两个独立分量。我们可以据此计算式 (140) 中每一项各自的渐近贡献。为简便起见，令 $n = n_t = n_r$ ，结果为

$$\left\{\frac{\delta\Gamma_{GR}}{\delta g_{tt}}, \frac{\delta\Gamma_{GR}}{\delta g_{rr}}\right\}\bigg|_{g=\bar{g}} \sim \frac{r^{-n-2}}{16\pi G_N}\left\{c_r\,(n-1)\,,(c_r - nc_t)\right\}, \tag{141a}$$

$$\frac{\delta\Gamma^{loc}_{\mathcal{R}^2}}{\delta g_{tt,rr}}\bigg|_{g=\bar{g}} \sim a_{tt,rr}r^{-n-4}, \tag{141b}$$

$$\frac{\delta\Gamma^{loc}_{\mathcal{R}^3}}{\delta g_{tt,rr}}\bigg|_{g=\bar{g}} \sim b_{tt,rr}r^{-8}, \tag{141c}$$

...(141d)

$$\frac{\delta\Gamma^{loc}_{\mathcal{R}^N}}{\delta g_{tt,rr}}\bigg|_{g=\bar{g}} \sim c_{tt,rr}r^{-2-6k-9l-2m}, \tag{141e}$$

where the last term is associated with operators of the form $\left[(\Delta^m R)\left(C^2\right)^k\left(C^3\right)^l\right]$ or other operators with different contractions and distributions of covariant derivatives. As anticipated, the coefficients $a_{tt,rr}, b_{tt,rr}$, and $c_{tt,rr}$, which we called a_j in Eq. (137), are functions of (n_i, c_i) and of the couplings in Γ_0 . As is evident from these expressions, if one would truncate the effective action Γ_0 to second order, disregarding terms $O\left(\mathcal{R}^3\right)$, then Γ_0 would not admit solutions with metric coefficients of the form (135). Indeed, there exists no value of n such that the contributions (141a) and (141b) to the field equations can cancel out. Including terms with six or more derivatives, a cancellation can occur for specific values of $n \geq 6$ and $n \neq 7$. For instance, including $\mathcal{R}^3$ operators yields the contribution (141c) to the field equations, which can cancel the general-relativity one for $n = 6$ (and for specific values of the $\mathcal{R}^3$ couplings [146]). To detail the robustness and generality of these results, it is key to understand the origin of the asymptotic scalings in Eq. (141). To this end we notice that:

其中最后一项对应形式为 $\left[(\Delta^m R)\left(C^2\right)^k\left(C^3\right)^l\right]$ 的算符，或其他具有不同协变导数缩并与分布的算符。正如预期，系数 $a_{tt,rr}, b_{tt,rr}$ 和 $c_{tt,rr}$ (即式 (137) 中我们记为 a_j 的量) 是 (n_i, c_i) 和 Γ_0 中耦合常数的函数。从上述表达式可以明显看出，如果将有效作用量 Γ_0 截断到二阶、忽略 $O\left(\mathcal{R}^3\right)$ 项，那么 Γ_0 不存在形式为 (135) 的度规系数解: 确实不存在 n 的取值能让场方程中 (141a) 和 (141b) 的贡献相互抵消。包含六阶或更高阶导数项后，对于特定取值的 $n \geq 6$ 和 $n \neq 7$ 可以实现抵消。例如，包含 $\mathcal{R}^3$ 算符会给场方程带来贡献 (141c)，当 $n = 6$ (且 $\mathcal{R}^3$ 的耦合常数取特定值 [146]) 时，该贡献可以抵消广义相对论的贡献。为了说明这些结果的稳健性与普遍性，理解式 (141) 中渐近标度的来源十分关键。为此我们注意到:

- Due to the nature of the asymptotic expansion of asymptotically flat spacetimes, all curvature tensors vanish at the expansion point $r \to \infty$. Asymptotically, every curvature tensor thus has to scale as r^{-h} with $h \neq 1$.

- 由于渐近平直时空的渐近展开性质，所有曲率张量在展开点 $r \to \infty$ 处均为零。因此，渐近区域内每个曲率张量都必须按 r^{-h} 标度，其中标度关系满足 $h \neq 1$ 。

- The more curvature tensors an operator in Γ_0 has, the more sub-leading its contribution to the field equations will be.

- Γ_0 中一个算符包含的曲率张量越多，它对场方程的贡献就越次要。

- Since the asymptotic scaling of the Ricci tensor contains r^{-n} , operators in Γ_0 including more than one occurrence of $R_{\mu\nu}$ will not allow to cancel the general-relativity contribution (141a) to the field equations for any $n > 0$.

- 由于里奇张量的渐近标度包含 r^{-n} ，因此 Γ_0 中出现多于一次 $R_{\mu\nu}$ 的算符无法抵消广义相对论对任意 $n > 0$ 场方程的贡献 (141a)。

- In order to cancel (141a) for a given n , one needs a leading-order scaling r^{-h} that is independent of n . Such a scaling can only come from operators built on Weyl tensors and at most one Ricci tensor or scalar.

- 要在给定 n 下抵消 (141a)，需要一个不依赖于 n 的领头阶标度 r^{-h} 。这种标度只能来自由外尔张量构造，且至多包含一个里奇张量或曲率标量的算符。

Since the first correction to the Schwarzschild scaling for Bardeen, Bonanno-Reuter, and Hayward black holes is $1/r^4$, i.e., $n = 4$, these solutions cannot be obtained from a principle of least action based on a local effective action. Enforcing these of other spacetimes with generic $1/r^n$ asymptotic scaling to stem from a principle of least action requires the inclusion of infrared nonlocalities, e.g., terms of the form $\mathcal{R}^2\Delta^{-\frac{n}{2}-2}\mathcal{R}$, in the effective action Γ_0 [146], as these operators contribute to the field equations with terms $\propto r^{-n-2}$ that can cancel the general-relativity scaling (141a) for any n . Yet, such nonlocalities would likely lead to observable deviations from GR. On the other hand, lapse functions displaying an exponential asymptotic scaling, such as those found in [42, 133, 134] and summarized in sections ”Self-Consistency and Iterative RG Improvement” and ”Effective Solutions from the Decoupling Mechanism,” may avoid this conclusion and be compatible with a principle of least action.

由于对于巴丁黑洞、博南诺-罗伊特黑洞和海沃德黑洞，其史瓦西标度的一阶修正为 $1/r^4$ ，即 $n = 4$ ，因此这些解无法从基于局域有效作用量的最小作用量原理得到。若要求这类具有通用 $1/r^n$ 渐近标度的其他时空满足最小作用量原理，则需要在有效作用量 Γ_0 中引入红外非局域性，例如 $\mathcal{R}^2\Delta^{-\frac{n}{2}-2}\mathcal{R}$ 形式的项 [146]，因为这类算符会给场方程贡献 $\propto r^{-n-2}$ 项，可在任意 n 下抵消广义相对论的标度项 (141a)。但这类非局域性很可能会导致可观测的偏离广义相对论的效应。另一方面，展现出指数渐近标度的时移函数，例如文献 [42, 133, 134] 中得到、并在“自洽性与迭代重整化群改进”和“去耦机制的有效解”两节中总结的结果，或许可以避开这一结论，与最小作用量原理相容。

Summary and Conclusions

总结与结论

The investigation of black holes in asymptotically safe gravity originated in the 1990s, with the seminal work by Bonanno and Reuter [48] on RG-improved black holes. Since then, the big picture of how black holes in an asymptotically safe universe should look like has evolved. In this book chapter, we provided a comprehensive chronology of this evolution, from the first works on RG-improved black holes and their generalizations (cf. section ”RG-Improved Black Holes”) to recent refinements of the method which attempt to remove its ambiguities (cf. section ”Improving the RG Improvement: Methods and Physical Results”) and more rigorous and solid findings grounded on first-principle calculations in gravitational effective field theory and quantum gravity (cf. section ”Toward Black Holes from First Principles”).

渐近平直安全引力中的黑洞研究起源于 20 世纪 90 年代，由博南诺与罗伊特完成了 RG 改进黑洞的开创性工作 [48]。自那以来，渐近安全宇宙中黑洞的整体图景不断发展。在本章中，我们全面梳理了这一发展的时间线，从 RG 改进黑洞及其推广的早期工作 (参见“RG 改进黑洞”一节)，到旨在消除方法歧义的最新改进 (参见“改进 RG 改进: 方法与物理结果”一节)，再到基于引力有效场论与量子引力第一性原理计算得到的更严谨可靠的结果 (参见“从第一性原理出发研究黑洞”一节)。

The RG improvement in gravity was introduced as a tool to account for leading-order quantum corrections to classical spacetimes and their dynamics. It consists in promoting the gravitational couplings to RG scale-dependent quantities and then identifying the artificial RG scale with a physical IR scale. Based on the decoupling mechanism [41], this procedure ought to provide a shortcut to (an approximation of) the quantum effective action [41] (cf. section ”RG Improvement: Key Idea”). While some of its most pressing drawbacks - including self-consistency [133], coordinate dependence [135], and scale setting [42,43,46,47,81,82] - have been partially addressed (cf. section ”Improving the RG Improvement: Methods and Physical Results”), the RG improvement in gravity is to be regarded as a model-building tool to construct quantum gravity-motivated models. Fully quantum solutions and their dynamics are to be derived from the quantum effective action.

引力中的 RG 改进最初是作为处理经典时空及其动力学领头阶量子修正的工具提出的。它的核心思路是将引力耦合推广为依赖 RG 标度的量，再将人工 RG 标度与物理红外标度对应起来。基于退耦机制 [41]，这一过程被认为是得到量子有效作用量 (近似) 的捷径 [41](参见“RG 改进: 核心思想”一节)。尽管 RG 改进的多个突出缺陷——包括自洽性 [133]、坐标依赖性 [135] 和标度设定 [42,43,46,47,81,82]——都已得到部分解决 (参见“改进 RG 改进: 方法与物理结果”一节)，它仍应被视为构建量子引力启发模型的建模工具，完整的量子解及其动力学需要从量子有效作用量导出。

In spite of its limitations, the RG improvement has allowed to build a number of asymptotic safety-inspired black holes (as well as cosmologies [124, 140, 167]) and to learn general lessons on black holes beyond general relativity (cf. section ”RG-Improved Black Holes”). Within the spherically symmetric, asymptotically flat setup, the first implementation of RG improvement yielded the Bonanno-Reuter class of solutions [48-50]. This is a family of regular black holes characterized by two horizons, akin to the well-known Dymnikova [134] and Hayward [88] solutions, whose evaporation process is believed to end up in a Planckian remnant. Among the variety of generalizations of Bonanno-Reuter black holes that have been put forth over the years [51-56,58,59,72,76,86], three ingredients are crucial in determining and testing realistic quantum-corrected black holes in asymptotic safety: cosmological constant, spin, and collapse dynamics. Their importance lies in that (i) the cosmological constant, in non-unimodular settings, might reintroduce the curvature singularity, unless the dimensionless cosmological constant vanishes at high energies and critical exponents satisfy certain bounds [74, 75]; (ii) non-rotating black holes are unlikely configurations, and, moreover, spin is decisive in establishing the shadow properties of black holes beyond general relativity [83-85]; and (iii) gravitational collapse renders singularity resolution less straightforward than in the static case, and it is likely that its endpoint be a black hole with an integrable singularity rather than a regular one [97-105]. In particular, this scenario might be desirable as it naturally avoids the potential perturbative instabilities characterizing regular black holes [64-71].

尽管存在局限性, RG 改进已经构建了多个受渐近安全启发的黑洞模型 (也包括宇宙学模型 [124, 140, 167])，并得到了广义相对论之外黑洞的一般性结论 (参见“RG 改进黑洞”一节)。在球对称渐近平直框架下，第一次 RG 改进实现得到了博南诺-罗伊特解族 [48-50]。这是一族具有双视界面的正则黑洞，与著名的迪姆尼科娃解 [134] 和海沃德解 [88] 类似，一般认为其蒸发过程最终会留下一个普朗克尺度残余。在多年来提出的各类博南诺-罗伊特黑洞推广工作 [51-56,58,59,72,76,86] 中，三个要素对确定和检验渐近安全下现实的量子修正黑洞至关重要: 宇宙学常数、自旋和坍缩动力学。它们的重要性在于:(i) 在非幺模框架下，宇宙学常数可能会重新引入曲率奇点，除非无量纲宇宙学常数在高能下消失，且临界指数满足一定边界条件 [74, 75]；(ii) 不旋转的黑洞是极不可能存在的构型, 此外，自旋对确定广义相对论之外黑洞的阴影性质起决定性作用 [83-85]；(iii) 引力坍缩使得奇点 resolved 比静态情况更难，其最终结果更可能是带有可积奇点的黑洞，而非正则黑洞 [97-105]。这种图景尤其更合理，因为它自然避免了正则黑洞特有的潜在微扰不稳定性 [64-71]。

A full understanding of quantum black holes within and beyond asymptotic safety requires going beyond RG improvement and deriving quantum corrections and theoretical constraints from top-down computations. In the last few years, much progress has been done in this direction, particularly concerning singularity-resolution mechanisms and general theoretical constraints (cf. section ”Toward Black Holes from First Principles”).

要完全理解渐近安全框架内外的量子黑洞，需要超越 RG 改进，从自上而下计算中导出量子修正和理论约束。近年来，这一方向已经取得了许多进展，特别是在奇点 resolve 机制和一般性理论约束方面 (参见“从第一性原理出发研究黑洞”一节)。

Singularity resolution within asymptotic safety can be related to the structure of the gravitational path integral [153, 154] or, equivalently, to the quantum effective action [34, 146]. As singularities result from the violent collapse of matter under its own gravity, their resolution could rely on an effective weakening of the gravitational interaction at high energies and short distances. Within asymptotic safety, such a weakening is understood in terms of the gravitational antiscreening associated with the Reuter fixed point [3] and may also reflect in scattering amplitudes that are everywhere finite in coordinate space [34]. At the level of the gravitational path integral, singularity resolution could be related to the suppression of singular configurations via the divergence of the corresponding on-shell action [153, 154]. Such a mechanism would require higher derivatives in the bare action.

渐近安全框架内的奇点 resolve 可以和引力路径积分 [153, 154] 的结构联系起来, 或者等价地说, 和量子有效作用量联系起来 [34, 146]。由于奇点源自物质在自身引力作用下的剧烈坍缩，奇点的 resolve 可能依赖于高能短距离下引力相互作用的有效弱化。在渐近安全中，这种弱化被解释为与罗伊特不动点相关的引力反屏蔽效应 [3]，也会体现为坐标空间中处处有限的散射振幅 [34]。在引力路径积分层面，奇点 resolve 可以通过裸作用量的散度压制奇异构型来实现 [153, 154]，这种机制要求裸作用量中存在更高导数项。

On top of singularity resolution, providing a fundamental, microscopic explanation of the black hole area law in asymptotic safety is crucial. First steps forward in this direction have been taken in [74, 147, 148].

Within the Einstein-Hilbert truncation, the scaling of the black hole entropy with the area comes from the structure of on-shell effective action: only boundary terms contribute to the entropy, as the bulk ones vanish on-shell. The validity of this mechanism beyond the Einstein-Hilbert truncation is so far unexplored, but if it will turn out to be stable in higher-order truncations - especially those leading to non-Ricci-flat solutions - it could provide a natural explanation for the holographic properties of gravity [168, 169].

除奇点消解之外，在渐近安全框架下为黑洞面积定律提供基础的微观解释也至关重要。该方向的初步研究已开展于文献 [74, 147, 148]。在爱因斯坦-希尔伯特截断下，黑洞熵随面积的标度关系来源于在壳有效作用量的结构: 仅边界项对熵有贡献，因为体项在壳上为零。该机制在爱因斯坦-希尔伯特截断外的有效性目前尚未得到探索，但如果证实它在高阶截断中稳定——尤其是在那些得到非里奇平坦解的截断中——它就能为引力的全息性质提供自然的解释 [168, 169]。

Singularity resolution, stability, and black hole entropy are not the only theoretical requirements constraining physical black holes beyond general relativity. While asymptotic deviations from classical black holes are expected to be tiny, and experimentally untestable, theoretical considerations can put surprisingly strong constraints on them [146]: requiring the validity of the principle of least action at large distances (i.e., imposing that a given metric is a solution to the dynamics stemming from a gravitational effective action) constrains the asymptotic scaling of black hole lapse functions beyond Schwarzschild and even allows to rule out some of the most popular alternatives to Schwarzschild black holes [146].

奇点消解、稳定性和黑洞熵并非约束广义相对论之外物理黑洞的唯一理论要求。虽然经典黑洞的渐近偏差预计极小，无法通过实验检验，但理论考量可以对其给出出人意料的强约束 [146]: 要求大距离下最小作用量原理成立 (即要求给定度规是引力有效作用量导出动力学的解)，会约束史瓦西解之外黑洞时移函数的渐近标度，甚至可以排除一些最受欢迎的史瓦西黑洞替代方案 [146]。

This highlights the prominent role of effective actions and the principle of least action in constraining quantum gravity-inspired models and driving theoretical investigations.

这凸显了有效作用量和最小作用量原理在约束量子引力启发模型、推动理论研究方面的重要作用。

Acknowledgments The author thanks J. Borissova, A. Held, and B. Knorr for comments on various sections of the book chapter, A. Held for many fruitful discussions on the derivations in [135], and B. Knorr for providing the numerical data of [34] to generate the plots in Fig. 13. A.P. acknowledges support by Perimeter Institute for Theoretical Physics. Research at Perimeter Institute is supported in part by the Government of Canada through the Department of Innovation, Science and Economic Development and by the Province of Ontario through the Ministry of Colleges and Universities. A.P. also acknowledges Nordita for support within the ”Nordita Distinguished Visitors” program and for hospitality during the last stages of development of this work. Nordita is supported in part by NordForsk.

致谢作者感谢 J. Borissova、A. Held 和 B. Knorr 对本章各部分内容提出的修改意见，感谢 A. Held 就文献 [135] 中的推导展开诸多富有成效的讨论，感谢 B. Knorr 提供文献 [34] 的数值数据以生成图 13 的图像。A.P. 感谢圆周理论物理研究所的支持。圆周研究所的研究部分得到加拿大政府通过创新、科学与经济发展部的资助，以及安大略省通过大学与学院厅的资助。A.P. 还感谢 Nordita 在“Nordita 杰出访问学者”项目下提供的支持，以及在本工作最终撰写阶段提供的招待。Nordita 部分得到 NordForsk 的资助。

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